it follows that the estimate of $CL_t \times IR_t$ is

$$cl_t \times ir_t = \frac{y_t}{tr_t \times sn_t}$$

Moreover, experience has shown that when considering either monthly or quarterly data, we can average out ir_t and thus calculate the estimate cl_t of CL_t by using

$$cl_t = \frac{cl_{t-1}ir_{t-1} + cl_t ir_t + cl_{t+1}ir_{t+1}}{3}$$

That is, cl_t is a three-period moving average of the $cl_t \times ir_t$ values.

Finally, we calculate the estimate ir_t or IR_t by using the equation

$$ir_t = \frac{cl_t \times ir_t}{cl_t}$$

The calculations of the values of cl_t and ir_t for the Tasty Cola data are summarized in Table 7.2(b). Since there are only three years of data, and since most of the values of cl_t are near 1, we cannot discern a well-defined cycle. Furthermore, in examining the values of ir_t, we cannot detect a pattern in the estimates of the irregular factors.

Traditionally, the estimates tr_t, sn_t, cl_t, and ir_t obtained by using the multiplicative decomposition method are used to describe the time series. However, we can also use these estimates to forecast future values of the time series. If there is no pattern in the irregular component, we predict that IR_t will equal one. Therefore, the point forecast of y_t is

$$\hat{y}_t = tr_t \times sn_t \times cl_t$$

if a well-defined cycle exists and can be predicted. The point forecast is

$$\hat{y}_t = tr_t \times sn_t$$

if a well-defined cycle does not exist or if CL_t cannot be predicted. For our Tasty Cola example, where

$$tr_t = b_0 + b_1 t = 380.163 + 9.489t$$

the point forecasts of the $n = 36$ historical Tasty Cola sales are given in Table 7.2(a). The point forecasts of future Tasty Cola sales in the 12 months of year 4 are as given in Table 7.4. For example, the point forecast of sales in period 44 is

$$\begin{aligned}\hat{y}_{44} &= tr_{44} \times sn_{44} \\ &= [380.163 + 9.489(44)](1.693) = 797.699(1.693) \\ &= 1350.50\end{aligned}$$

Although there is no theoretically correct prediction interval for y_t, the authors have found that a fairly accurate (approximate) $100(1 - \alpha)\%$ prediction interval for y_t is

$$[\hat{y}_t \pm B_t[100(1 - \alpha)]]$$

TABLE 7.4 Forecasts of Future Values of Tasty Cola Sales Calculated Using Multiplicative Decomposition

t	sn_t	$\text{tr}_t = 380.163 + 9.489t$	$\hat{y}_t = \text{tr}_t \times \text{sn}_t$	$B_t(95)$	$[\hat{y}_t - B_t(95), \hat{y}_t + B_t(95)]$	y_t
37	.493	731.273	360.52	26.80	[333.72, 387.32]	352
38	.596	740.762	441.48	26.92	[414.56, 468.40]	445
39	.595	750.252	446.40	27.04	[419.36, 473.44]	453
40	.680	759.741	516.62	27.17	[489.45, 543.79]	541
41	.564	769.231	433.85	27.30	[406.55, 461.15]	457
42	.986	778.720	767.82	27.44	[740.38, 795.26]	762
43	1.467	788.209	1156.30	27.59	[1128.71, 1183.89]	1194
44	1.693	797.699	1350.50	27.74	[1322.76, 1378.24]	1361
45	1.990	807.188	1606.30	27.89	[1578.41, 1634.19]	1615
46	1.307	816.678	1067.40	28.05	[1039.35, 1095.45]	1059
47	1.029	826.167	850.12	28.22	[821.90, 878.34]	824
48	.600	835.657	501.39	28.39	[473, 529.78]	495

where $B_t[100(1-\alpha)]$ is the error bound in a $100(1-\alpha)\%$ prediction interval

$$[\text{tr}_t \pm B_t[100(1-\alpha)]]$$

for the deseasonalized observation

$$\begin{aligned} d_t &= \text{TR}_t + \varepsilon_t \\ &= \beta_0 + \beta_1 t + \varepsilon_t \end{aligned}$$

For example, using SAS to predict d_t on the basis of t by using the above trend line, we find that a 95% prediction interval for d_{44} is

$$[769.959, 825.439]$$

This implies that

$$\begin{aligned} B_{44}[95] &= \frac{825.439 - 769.959}{2} \\ &= 27.74 \end{aligned}$$

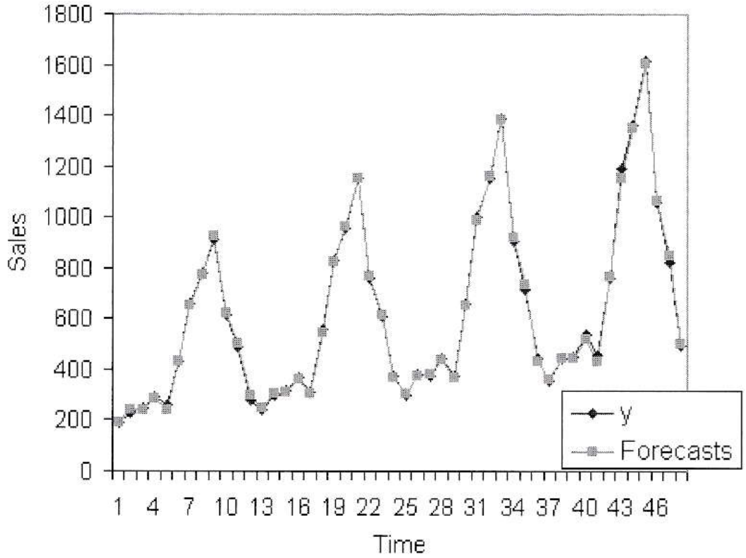
It follows that an approximate 95% prediction interval for y_{44} is

$$[1350.50 - 27.74, 1350.50 + 27.74] = [1322.76, 1378.24]$$

In Table 7.4 we present 95% prediction intervals (calculated by the above method) for Tasty Cola sales in the 12 months of year 4.

Suppose that we actually observe Tasty Cola sales in year 4 and that these sales are as given in Table 7.4. In Figure 7.4 we plot the observed and forecasted sales for all 48 sales periods. In practice the comparison of the observed and forecasted sales in years 1 through 3 would be used by the analyst to determine whether the forecasting equation adequately fits the historical data. An adequate fit (as indicated by Figure 7.4, for example) might prompt an analyst to use this equation to calculate forecasts for future time periods.

FIGURE 7.4 Forecasts of historical and future values of Tasty Cola sales calculated using multiplicative decomposition



One reason that the Tasty Cola forecasting equation

$$\begin{aligned}\hat{y}_t &= \text{tr}_t \times \text{sn}_t \\ &= (380.163 + 9.489t)\text{sn}_t\end{aligned}$$

provides reasonable forecasts is that this equation *multiplies* sn_t by tr_t. Therefore, as the average level of the time series (determined by the trend) increases, the seasonal swing of the time series increases, which is consistent with the data plots in Figures 7.2 and 7.4. For example, note from Table 7.3 that the estimated seasonal factor for August is 1.693. The forecasting equation yields a prediction of Tasty Cola sales in August of year 1 equal to

$$\begin{aligned}\hat{y}_8 &= [380.163 + 9.489(8)](1.693) \\ &= (456.075)(1.693) \\ &= 772.13\end{aligned}$$

This implies a seasonal swing of 772.13 − 456.075 = 316.055 (hundreds of cases) above 456.075, the estimated trend. The forecasting equation yields a prediction of Tasty Cola sales in August of year 2 equal to

$$\begin{aligned}\hat{y}_{20} &= [380.163 + 9.489(20)](1.693) \\ &= (569.943)(1.693) \\ &= 964.91\end{aligned}$$

which implies an *increased* seasonal swing of 964.91 − 569.943 = 394.967 (hundreds of cases) above 569.943, the estimated trend. In general, then, the forecasting equation is appropriate for forecasting a time series with a seasonal swing that is proportional to the average level of the time series as determined by the trend—that is, a time series exhibiting increasing seasonal variation. In fact, sometimes increasing seasonal variation is referred to as **multiplicative seasonal variation.**

7.2 ADDITIVE DECOMPOSITION

Consider a time series that exhibits constant seasonal variation. When the parameters describing the series are not changing over time, the time series can sometimes be modeled adequately by using what is called the **additive decomposition model.**

The **additive decomposition model** is

$$y_t = \text{TR}_t + \text{SN}_t + \text{CL}_t + \text{IR}_t$$

Here TR_t, SN_t, CL_t, and IR_t are again defined to be, respectively, trend, seasonal, cyclical, and irregular factors. However, in this case these factors are additive rather than multiplicative.

The **additive decomposition method** can be used to obtain point estimates tr_t, sn_t, cl_t, and ir_t of the above factors. The procedure begins with the calculation of centered moving averages, CMA_t. The centered moving average is regarded as an estimate of $\text{TR}_t + \text{CL}_t$. Since the model

$$y_t = \text{TR}_t + \text{SN}_t + \text{CL}_t + \text{IR}_t$$

implies that

$$\text{SN}_t + \text{IR}_t = y_t - (\text{TR}_t + \text{CL}_t)$$

it follows that the estimate $\text{sn}_t + \text{ir}_t$ of $\text{SN}_t + \text{IR}_t$ is

$$\text{sn}_t + \text{ir}_t = y_t - (\text{tr}_t + \text{cl}_t) = y_t - \text{CMA}_t$$

In order to obtain sn_t, we group the values of $\text{sn}_t + \text{ir}_t$ by like seasons (months, quarters, etc., as appropriate). For each season we compute the average $\overline{\text{sn}}_t$ of the $\text{sn}_t + \text{ir}_t$ values for that season. We obtain seasonal factors by normalizing the $\overline{\text{sn}}_t$ values so that the normalized values sum to zero. The normalization is accomplished by subtracting the quantity $\Sigma_{t=1}^{L}\overline{\text{sn}}_t/L$ from each of the $\overline{\text{sn}}_t$ values. That is, the estimate of SN_t is

$$\text{sn}_t = \overline{\text{sn}}_t - \left(\sum_{t=1}^{L} \overline{\text{sn}}_t / L\right)$$

We next calculate the deseasonalized observation in time period t to be

$$d_t = y_t - \text{sn}_t$$

Subtracting sn_t from the observation y_t removes the seasonality from the data and allows us to estimate the trend better. We obtain the estimate tr_t of the trend TR_t by fitting a regression equation to the deseasonalized data. For example, a linear trend

$$\text{TR}_t = \beta_0 + \beta_1 t$$

or a quadratic trend

$$\text{TR}_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

might be fitted to the deseasonalized observations.

Since the model

$$y_t = \text{TR}_t + \text{SN}_t + \text{CL}_t + \text{IR}_t$$

implies that

$$\text{CL}_t + \text{IR}_t = y_t - \text{TR}_t - \text{SN}_t$$

it follows that we compute the estimate of $\text{CL}_t + \text{IR}_t$ to be

$$\text{cl}_t + \text{ir}_t = y_t - \text{tr}_t - \text{sn}_t$$

In order to average out ir_t, we compute a three-period moving average of the $\text{cl}_t + \text{ir}_t$ values. That is, the estimate of CL_t is

$$\text{cl}_t = \frac{(\text{cl}_{t-1} + \text{ir}_{t-1}) + (\text{cl}_t + \text{ir}_t) + (\text{cl}_{t+1} + \text{ir}_{t+1})}{3}$$

Finally, we calculate the estimate of IR_t to be

$$\text{ir}_t = (\text{cl}_t + \text{ir}_t) - \text{cl}_t$$

The estimates tr_t, sn_t, cl_t, and ir_t are generally used to describe the time series. We can also use these estimates to compute predictions. If there is no pattern in the irregular component, we predict IR_t to equal zero. It follows that the point forecast of y_t is

$$\hat{y}_t = \text{tr}_t + \text{sn}_t + \text{cl}_t$$

if a well-defined cycle exists and can be predicted. The point forecast is

$$\hat{y}_t = \text{tr}_t + \text{sn}_t$$

if no well-defined cycle exists or if CL_t cannot be predicted. Although there is no theoretically correct prediction interval for y_t, an approximate $100(1 - \alpha)\%$ prediction interval for y_t is

$$[\hat{y}_t \pm B_t[100(1 - \alpha)]]$$

where $B_t[100(1 - \alpha)]$ is the error bound in a $100(1 - \alpha)\%$ prediction interval

$$[\text{tr}_t \pm B_t[100(1 - \alpha)]]$$

for the deseasonalized observation $d_t = y_t - \text{sn}_t$.