UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611

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LINEAR ALGEBRA

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100 Marks

EXAMINERS .
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This paper consists of 3 pages

ANSWER ALL THE QUESTIONS.

QUESTION 1

Suppose that A is a square matrix with characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$

(a) What is the order of A? (1)

(b) Is A invertible? (1)

(c) Is A diagonalizable? (2)

(d) Find the eigenvalues of A^2 (1)

Justify your answers. **[5]**

QUESTION 2

Let

$$A = \begin{bmatrix} -3 & 1 & 0 \\ -6 & 2 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and show that the eigenvalues of A are 0 and -1 (4)

(b) Find a basis for each eigenspace of A (6)

(c) Explain why A is diagonalizable (1)

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ (3)

(e) Use (d) to calculate A^{99} (5)

[19]

[TURN OVER]

QUESTION 3

Explain in each case whether $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a linear transformation. If it is, supply a proof, if not, supply a counterexample.

(a) $T(a, b) = a + b$ (4)

(b) $T(a, b) = ab$ (4)

[8]

QUESTION 4

Let $T : P_2 \longrightarrow P_2$ be the linear transformation defined by

$$\begin{aligned} T : P_2 &\longrightarrow P_2 \\ T(p(x)) &= p(x-1) \end{aligned}$$

and let

$$B = \{1, x, x^2\} \quad \text{and} \quad B' = \{1 + x + x^2, 2x + x^2, x + x^2\}$$

(a) Find the matrix of T with respect to B. (3)

(b) Verify that $[T]_B[q]_B = [T(q)]_B$ for each $q(x) = c_0 + c_1x + c_2x^2$ in P_2. (4)

(c) Find the transition matrix P from B' to B. (2)

(d) Write down the formula for $[T]_{B'}$ in terms of $[T]_B$ and P, and use it to calculate $[T]_{B'}$. (9)

[18]

QUESTION 5

(a) Let W be a subset of a vector space V. When is W a *subspace* of V? (3)

(b) Let $T : V \longrightarrow W$ be a linear transformation.

(i) Define $\ker(T)$, the *kernel* of T. (1)

(ii) Show that $\ker(T)$ is a subspace of V. (5)

(iii) Prove that T is one-to-one if and only if the kernel of T contains only the zero vector, i.e. $\ker(T) = \{\mathbf{0}\}$.

(6)

[15]

[TURN OVER]

QUESTION 6

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 2 \end{bmatrix}$

(a) Find a base for the nullspace of A (6)

(b) State and use an applicable theorem to determine the rank of A (4)

[10]

QUESTION 7

Consider the bases $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathbb{R}^2$ where

$$\begin{aligned} S &: \mathbf{v}_1 = (1,0), \mathbf{v}_2 = (0,1) \\ T & \quad \mathbf{w}_1 = (1,1), \mathbf{w}_2 = (2,1). \end{aligned}$$

(a) Find the transition matrix $P_{S\longleftarrow T}$ from the T-base to the S-base. (4)

(b) Find the transition matrix $Q_{T\longleftarrow S}$ from the S-base to the T-base (4)

(c) Use your answer in (a) to find $[v]_S$ if $[v]_T = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ (2)

[10]

QUESTION 8

(a) Let $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ be polynomials in P_2. Show that

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1)$$

is an inner product on P_2. (8)

(b) Let $\mathbb{R}^3$ have the Euclidean inner product. Find an *orthonormal* basis for the subspace spanned by $\{(0,1,2), (-1,0,1)\}$. (7)

[15]

TOTAL: [100]