MAT2611 October/November 2010 Exam Solutions

Q1 From the characteristic polynomial $p(\lambda) = \lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1)$ we find that A has the eigenvalues 0, 1 and -1.

(a) The order of A is 3 (degree of $p(\lambda)$).

(b) No, since 0 is an eigenvalue of A.

(c) Yes, since the eigenvalues are all different.

(d) The eigenvalues of A^2 are $0^2 = 0$, $1^2 = 1$ and $(-1)^2 = 1$.

Q2 (a) The characteristic polynomial of A in λ is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & -1 & 0 \\ 6 & \lambda-2 & 0 \\ 3 & -1 & \lambda \end{vmatrix} = (\lambda+3)(\lambda-2)\lambda + 6\lambda = \lambda^2(\lambda+1)$$

so that the eigenvalues of A are 0 (twice) and -1.

(b) First we consider the eigenvalue 0. Thus we solve

$$0I - A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix} 3 & -1 & 0 & : & 0 \\ 6 & -2 & 0 & : & 0 \\ 3 & -1 & 0 & : & 0 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 3 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

so that $3x = y$. Thus the eigenspace is

$$\left\{ \begin{bmatrix} x \\ 3x \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : x, z \in \mathbb{R} \right\}$$

A basis for the eigenspace corresponding to the eigenvalue 0 is given by

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Don't forget to verify that these vectors are linearly independent. If they are not, use row reduction of the matrix with these vectors **as rows** and the non-zero **rows** of the matrix will form a basis (i.e. rewrite the non-zero rows as column vectors).

(c) We found 3 linearly independent eigenvectors of A.

(d) We construct P directly from the eigenvectors we found:

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

then the eigenvalues in D appear in the same order as the corresponding eigenvectors

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now check that $P^{-1}AP = D$ (exercise).

(e) From $P^{-1}AP = D$ we find

$$P^{-1}A^{99}P = D^{99} = \begin{bmatrix} 0^{99} & 0 & 0 \\ 0 & 0^{99} & 0 \\ 0 & 0 & (-1)^{99} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

so that $P^{-1}A^{99}P = D$ and $A^{99} = PDP^{-1} = A$.

Q3 (a) Let $\mathbf{x} = (a, b), \mathbf{y} = (\alpha, \beta) \in \mathbb{R}^2$ with $k, a, b, \alpha, \beta \in \mathbb{R}$. Now

$$\begin{aligned} T(k\mathbf{x}) &= T(ka, kb) = (ka) + (kb) = k(a + b) = kT(a, b) = kT(\mathbf{x}) \\ T(\mathbf{x} + \mathbf{y}) &= T(a + \alpha, b + \beta) = (a + \alpha) + (b + \beta) = (a + b) + (\alpha + \beta) \\ &= T(a, b) + T(\alpha, \beta) = T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

so that T is linear.

(b) Let $\mathbf{x} = (a, b) \in \mathbb{R}^2$ with $k, a, b \in \mathbb{R}$. Now

$$T(k\mathbf{x}) = T(ka, kb) = (ka)(kb) = k^2(ab) = k^2T(a, b) = k^2T(\mathbf{x}).$$

Now consider $a = b = 1$ and $k = 2$, then $T(k\mathbf{x}) = 4$ and $kT(\mathbf{x}) = 2$. This provides a counterexample for $T(k\mathbf{x}) = kT(\mathbf{x})$. Thus T is not linear.

Q4 (a) Applying T to the basis B we find

$$\begin{aligned} T(1) &= 1\Big|_{x \to x-1} = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) &= x\Big|_{x \to x-1} = x - 1 = -1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ T(x^2) &= x^2\Big|_{x \to x-1} = (x - 1)^2 = x^2 - 2x + 1 = 1 \cdot 1 - 2 \cdot x + 1 \cdot x^2 \end{aligned}$$

and writing the coefficients of the basis vectors for each equation as the column vectors in the matrix representation we find

$$[T]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) We have

$$T(q(x)) = c_0 + c_1x + c_2x^2\Big|_{x \to x-1} = c_0 + c_1(x-1) + c_2(x-1)^2 = (c_0 - c_1 + c_2) + (c_1 - 2c_2)x + c_2x^2$$

so that

$$[T(q(x))]_B = \begin{bmatrix} c_0 - c_1 + c_2 \\ c_1 - 2c_2 \\ c_2 \end{bmatrix}.$$

We also have

$$[q(x)]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

so that

$$[T]_B[q(x)]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_0 - c_1 + c_2 \\ c_1 - 2c_2 \\ c_2 \end{bmatrix} = [T(q(x))]_B.$$

(c) We must express each element of B' in terms of the elements of B, i.e. set $b(x) = \alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2$ and solve for $\alpha, \beta, \gamma \in \mathbb{R}$ for each $b(x) \in B'$. In this case the calculation is straightforward:

$$\begin{aligned} 1 + x + x^2 &= 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2 \\ 2x + x^2 &= 0 \cdot 1 + 2 \cdot x + 1 \cdot x^2 \\ x + x^2 &= 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 \end{aligned}$$

Each equation's coefficients provides the columns for the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(d) To calculate T in the basis B' we first convert B' to B using P, then we apply $[T]_B$ in B, and then convert back from B to B' using P^{-1}:

$$[T]_{B'} = P_{B' \leftarrow B}[T]_B P_{B \leftarrow B'} = P^{-1}[T]_B P.$$

We apply row-reduction to the matrix P augmented with the identity to find P^{-1}.

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 1 & 2 & 1 & : & 0 & 1 & 0 \\ 1 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \sim \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 2 & 1 & : & -1 & 1 & 0 \\ 0 & 1 & 1 & : & -1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \sim \\ R_2 - R_3 \\ \sim \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & -1 \\ 0 & 1 & 1 & : & -1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \sim \\ R_3 - R_2 \\ \sim \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & -1 \\ 0 & 0 & 1 & : & -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

Thus (exercise: check that $P^{-1}P = PP^{-1} = I$)

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Now

$$\begin{aligned} [T]_{B'} = P^{-1}[T]_B P &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & -1 & -2 \\ 2 & 3 & 3 \end{bmatrix}. \end{aligned}$$

Q5 (a)
- * W must be non-empty
- * For all $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \in W$ (closure under vector addition)
- * For all $k \in \mathbb{R}$ and $\mathbf{x} \in W$, $k\mathbf{x} \in W$ (closure under scalar multiplication)

(b) Let $\mathbf{0}_V$ be the zero vector in V and $\mathbf{0}_W$ be the zero vector in W.

(i) The kernel of T is

$$\ker(T) := \{\, \mathbf{v} \in V \;:\; T(\mathbf{v}) = \mathbf{0}_W \,\}.$$

(ii) · Since T is linear and $0\mathbf{x} = \mathbf{0}_W$ for all $\mathbf{x} \in W$ we have

$$T(\mathbf{0}_V) = T(0\mathbf{y}) = 0T(\mathbf{y}) = \mathbf{0}_W$$

for all $\mathbf{y} \in V$. Thus $\mathbf{0}_V \in \ker(T)$, and $\ker(T)$ is non-empty.

· Let $k \in \mathbb{R}$ and $\mathbf{a} \in \ker(T)$. Then $T(k\mathbf{a}) = kT(\mathbf{a}) = k\mathbf{0}_W = \mathbf{0}_W$. Thus $k\mathbf{a} \in \ker(T)$.

· Let $\mathbf{a}, \mathbf{b} \in \ker(T)$. Then $T(\mathbf{a} + \mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$. Thus $\mathbf{a} + \mathbf{b} \in \ker(T)$.

(iii) T is one-to-one
if and only if
$\forall \mathbf{x}, \mathbf{y} \in V\colon\ T(\mathbf{x}) = T(\mathbf{y}) \Leftrightarrow \mathbf{x} = \mathbf{y}$
if and only if
$\forall \mathbf{x}, \mathbf{y} \in V\colon\ T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{0}_W \Leftrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0}_V$
if and only if
$\forall \mathbf{x}, \mathbf{y} \in V\colon\ T(\mathbf{x} - \mathbf{y}) = \mathbf{0}_W \Leftrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0}_V$
if and only if
$\forall \mathbf{z} \in V\colon\ T(\mathbf{z}) = \mathbf{0}_W \Leftrightarrow \mathbf{z} = \mathbf{0}_V$
if and only if
$\ker(T) = \{\, \mathbf{0}_V \,\}$; where we used the linearity of T and substituted $\mathbf{z} := \mathbf{x} - \mathbf{y}$.

Q6 (a) We solve

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$. Applying row-reduction we find

$$\begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 1 & 0 & 1 & : & 0 \\ 2 & -2 & 2 & : & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & : & 0 \\ 1 & 0 & 1 & : & 0 \\ 0 & -2 & 0 & : & 0 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ R_3 + 2R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

so that $c = -a$ and $b = 0$. Thus

$$\text{nullspace(A)} = \left\{ \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} : a \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

Since there is only one free parameter we easily find the basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

for nullspace(A).

(b) Since $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3$ (the number of columns of A) we find

$$\operatorname{rank}(A) = 3 - \operatorname{nullity}(A) = 3 - 1 = 2.$$

Q7 (a) We solve

$$\mathbf{w}_1 = a\mathbf{v}_1 + b\mathbf{v}_2, \qquad \mathbf{w}_2 = c\mathbf{v}_1 + d\mathbf{v}_2$$

for $a, b, c, d \in \mathbb{R}$ to find $a = b = d = 1$ and $c = 2$. We have

$$P_{S\leftarrow T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b) We solve

$$\mathbf{v}_1 = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2, \qquad \mathbf{v}_2 = \gamma\mathbf{w}_1 + \delta\mathbf{w}_2$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ to find $\alpha = -1$, $\beta = 1$, $\gamma = 2$ and $\delta = -1$. We have

$$Q_{T\leftarrow S} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Notice that $Q_{T\leftarrow S} = P^{-1}_{S\leftarrow T}$ which provides another way to find $Q_{T\leftarrow S}$.

(c) We have

$$[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

Q8 (a) Let $k \in \mathbb{R}$ and $\mathbf{p} = p(x), \mathbf{q} = q(x), \mathbf{r} = r(x) \in P_2$.

$*$ $\langle \mathbf{q}, \mathbf{p} \rangle = q(0)p(0) + q\left(\frac{1}{2}\right)p\left(\frac{1}{2}\right) + q(1)p(1)$
$= p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1) = \langle \mathbf{p}, \mathbf{q} \rangle$

$*$ $\langle k\mathbf{p}, \mathbf{q} \rangle = (kp)(0)q(0) + (kp)\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + (kp)(1)q(1)$
$= k\left(p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1)\right) = k\langle \mathbf{p}, \mathbf{q} \rangle$

$*$ $\langle \mathbf{p} + \mathbf{q}, \mathbf{r} \rangle = (p+q)(0)r(0) + (p+q)\left(\frac{1}{2}\right)r\left(\frac{1}{2}\right) + (p+q)(1)r(1)$
$= p(0)r(0) + q(0)r(0) + p\left(\frac{1}{2}\right)r\left(\frac{1}{2}\right) + q\left(\frac{1}{2}\right)r\left(\frac{1}{2}\right) + p(1)r(1) + q(1)r(1)$
$= p(0)r(0) + p\left(\frac{1}{2}\right)r\left(\frac{1}{2}\right) + p(1)r(1) + q(0)r(0) + q\left(\frac{1}{2}\right)r\left(\frac{1}{2}\right) + q(1)r(1)$
$= \langle \mathbf{p}, \mathbf{r} \rangle + \langle \mathbf{q}, \mathbf{r} \rangle$.

$*$ $\langle \mathbf{p}, \mathbf{p} \rangle = p(0)p(0) + p\left(\frac{1}{2}\right)p\left(\frac{1}{2}\right) + p(1)p(1) = p(0)^2 + p\left(\frac{1}{2}\right)^2 + p(1)^2 \geq 0$
Now let $\mathbf{p} = p(x) = p_0 + p_1x + p_2x^2 \in P_2$ where $p_0, p_1, p_2 \in \mathbb{R}$.
Then $\langle \mathbf{p}, \mathbf{p} \rangle = p_0^2 + \left(p_0 + \frac{p_1}{2} + \frac{p_2}{4}\right)^2 + (p_0 + p_1 + p_2)^2 = 0$
if and only if $p_0 = p_0 + \frac{p_1}{2} + \frac{p_2}{4} = p_0 + p_1 + p_2 = 0$
if and only if $p_0 = 0, 2p_1 + p_2 = 0, p_1 + p_2 = 0$
if and only if $p_0 = 0, p_1 = 0, p_2 = 0$
where we subtracted the last equation from the middle equation.

(b) We apply the Gram-Schmidt process

$$\mathbf{v}_1 = (0, 1, 2)$$

$$\mathbf{v}_2 = (-1, 0, 1) - \frac{\langle \mathbf{v}_1, (0,1,2) \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (-1, 0, 1) - \frac{2}{5}(0, 1, 2)$$

$$= \left(-1, -\frac{2}{5}, \frac{1}{5}\right) = \frac{1}{5}(-5, -2, 1).$$

Normalizing v_1 and v_2 we find the orthonormal basis

$$\left\{ \frac{\mathbf{v}_1}{\sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}}, \frac{\mathbf{v}_2}{\sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}} \right\} = \left\{ \frac{\mathbf{v}_1}{\sqrt{5}}, \frac{\mathbf{v}_2}{\frac{\sqrt{30}}{5}} \right\} = \left\{ \frac{1}{\sqrt{5}}(0, 1, 2), \frac{1}{\sqrt{30}}(-5, -2, 1) \right\}.$$