UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611

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LINEAR ALGEBRA

100 Marks

Duration 2 Hours

EXAMINERS :
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This paper consists of 3 pages

ANSWER ALL THE QUESTIONS.

QUESTION 1

Let V be a vector space and $B = \{b_1, b_2, \ldots, b_n\} \subseteq V$.

(1.1) Define what it means to say that

(a) B is *linearly independent*, (2)

(b) B *spans* V (2)

(1.2) When is V said to be a *finite-dimensional* vector space? (1)

(1.3) What conditions must B satisfy in order for it to be a *subspace* of V? (3)

(1.4) Let $p(x) = 1 - x, q(x) = 5 + 3x - 2x^2$ and $r(x) = 1 + 3x - x^2$

(a) Determine whether $B = \{p(x), q(x), r(x)\}$ is a linearly independent subset of the vector space P_2. (6)

(b) Is B a basis for P_2? Explain (1)

(1.5) Let $S = \{(x, -x, 0) \quad x \in \mathbb{R}\} \subseteq \mathbb{R}^3$

(a) Show that S is a subspace of $\mathbb{R}^3$. (6)

(b) Determine a basis for S. (2)

(c) What is the dimension of S? Explain your answer. (2)

[25]

[TURN OVER]

QUESTION 2

(2 1) Let

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

(a) Find a basis for the nullspace of A. **(10)**

(c) Hence, determine the rank of A using an applicable theorem **(3)**

(2 2) Find an orthonormal basis for the subspace V of $\mathbb{R}^2$ with basis $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$. **(8)**

(2 3) Explain whether the matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

is orthogonal **(4)**

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QUESTION 3

Let A be an $n \times n$ matrix and suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$.

(3 1) Define what is meant by the

(a) *algebraic multiplicity* of λ; **(2)**

(b) *geometric multiplicity* of λ **(2)**

(3 2) Which of the two multiplicities of λ in (3.1) is always greater than or equal to the other? **(1)**

(3 3) Prove that A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A **(6)**

(3 4) Let

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}.$$

Find a matrix P which diagonalizes A, and write down the corresponding diagonal matrix D **(11)**

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[TURN OVER]

QUESTION 4

Let $T : P_1 \to \mathbb{R}^2$ be the function defined by

$$T(a+bx) = A\begin{bmatrix} a \\ b \end{bmatrix},$$

where A is a fixed 2×2 matrix

(4.1) Show that T is a linear transformation (5)

(4.2) For $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$,

(a) show that T is one-to-one; (3)

(b) find $T^{-1}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$. (4)

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QUESTION 5

Let $B = \{1, x, x^2\}$ and $B' = \{1, 1+x, (1+x)^2\}$ be bases for P_2. The transition matrix from the B-basis to the B'-basis is given by

$$P_{B' \leftarrow B} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $T : P_2 \to P_2$ be the linear transformation such that

$$[T]_{B'} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(5.1) Find the transition matrix $P_{B \leftarrow B'}$ from the B'-basis to the B-basis. (3)

(5.2) Use $P_{B' \leftarrow B}$ to express $p(x) = a + bx + cx^2$ as a linear combination of the vectors in B'. (3)

(5.3) Find $[T]_B$ (6)

(5.4) Find $[T(a + bx + cx^2)]_B$ (3)

(5.5) Find $T(a + bx + cx^2)$ (1)

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TOTAL MARKS: [100]