UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611 (489569) October/November 2017

LINEAR ALGEBRA

Duration 2 Hours

105 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT

Use of a non-programmable pocket calculator is permissible

Closed book examination.

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This paper consists of 18 pages

Instructions

- The notation is the same as in the Study Guide
- Answer all questions
- All calculations must be shown
- Pages 1-8 contain the four questions for this examination
- Pages 9 - 18 contain all the definitions and statements of theorems necessary for this examination You are allowed to refer to this material **only** and no other **external material** for this examination
- Quesion 1 consists of multiple-choice questions and should be answered on the multiple choice answer sheet
- Answer Questions 2-4 in your answer book

[TURN OVER]

Question 1.

This question consists entirely of <u>multiple choice questions</u> and should be answered in the <u>MCQ sheet</u>.
Each question has only **<u>one</u>** *correct answer. Write only the* **<u>number</u>** *for your answer (i.e., 1, 2, 3, 4 or 5).*

1.1 On the set of real numbers $\mathbb{R}$, define two operations $\oplus$ (addition) and $\circ$ (multiplication) as follows

$$a \oplus b = a + b - 1, \text{ for all } a, b \in \mathbb{R},$$

and

$$a \circ b = ab - a + 1, \text{ for all } a, b \in \mathbb{R}$$

Then $\mathbb{R}$ is a vector space over itself with respect to $\oplus$ and $\circ$.
Which statement about the zero $\mathbf{z}$ of $\mathbb{R}(\oplus, \circ)$ in the following is true

1. $\mathbf{z} = 0$
2. $\mathbf{z} = 1$
3. $\mathbf{z} = 4$
4. $\mathbf{z} = 2$
5. None of the above

[3 marks]

1.2 Which of the following are subspaces of P_2

A span $\{1, x\}$

B $\{2 + \frac{x}{3}\}$

C $\{a + ax^2 : a \in \mathbb{R}\}$

D span $\{1 + x^3\}$

Select from the following

1. A only
2. A and B
3. A and C
4. C and D
5. None of the above

[3 marks]

1 3. Which of the following sets of vectors are linearly independent

A. $\{(1,2,0),(1,0,1),(1,0,0)\} \subseteq \mathbb{R}^3$

B $\{(\alpha,2\alpha,0),(1,0,1),(1,0,0)\} \subseteq \mathbb{R}^3$, $\alpha \neq 0$

C $\left\{\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 1\end{bmatrix}\right\}$

Select from the following

1. Only A
2 Only C
3 A and B only
4. B and C only
5 None of the above

[3 marks]

1.4. Let X be the subspace of M_{22} defined as·

$$X = \left\{\begin{bmatrix}a & 0\\b & a\end{bmatrix} : a, b \in \mathbb{R}\right\}.$$

Which of the following sets form a basis for X:

A $\left\{\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\2 & -1\end{bmatrix}\right\}$

B. $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$

C $\left\{\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, \begin{bmatrix}0 & 0\\-2 & 0\end{bmatrix}\right\}$

Select from the following:

1. A and B
2. A and C
3 C only
4 B and C
5. None of the above

[3 marks]

[TURN OVER]

1 5 Which of the following statements are true

A $\dim \operatorname{span}\{(1,1,-1),(1,0,1)\} = 2$ in $\mathbb{R}^3$

B $\dim \operatorname{span}\{1-x, x^2+x\} = 2$ in P_2

C $\dim \operatorname{span}\left\{\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\right\} = 2$ in M_{22}

Select from the following

1 A only

2 A and C only

3 A and B

4 A, B and C

5 None of the above

[3 marks]

1 6 It is given that the rank of

$$\begin{bmatrix}\alpha & 1 & -1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}$$

is not equal to 3

Then the value of α is

1 2

2 1

3 4

4 -1

5 None of the above

[3 marks]

1 7 Which of the following sets form a basis for the column space of $\begin{bmatrix}2 & 3\\ 0 & 0\\ -1 & -1\end{bmatrix}$?

A $\{[2,0,-1]^{\mathbf{T}}, [-3,0,1]^{\mathbf{T}}\}$

B $\{(2,3),(0,0),(-1,-1)\}$

C $\{[2,0,-1]^{\mathbf{T}}\}$

D $\{[-4,0,2]^{\mathbf{T}}, [3,0,-1]^{\mathbf{T}}\}$

Select from the following

1 A and D only
2. A only
3 A and B only
4. C only
5 None of the above.

[3 marks]

1.8 Which of the following sets are a basis for the row space of:

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

A $\{(-1,1,1),(0,1,1)\}$
B $\{(1,1,1),(0,1,1)\}$
C. $\{(2,1,1),(0,3,3)\}$

Select from the following:

1. A only
2. A and B only
3. A and C only
4 C only
5 None of the above

[3 marks]

1 9. Which of the following sets are a basis for the null space of

$$\begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

A $\{[0,1,-1]^{\mathbf{T}},[0,0,0]^{\mathbf{T}}\}$
B. $\{[0,1,-1]^{\mathbf{T}}\}$
C. $\{[0,2,-2]^{\mathbf{T}}\}$
D. $\{[2,-1,0]^{\mathbf{T}},[0,1,-1]^{\mathbf{T}}\}$

Select from the following

1. B and C only
2. A only
3 B only

4 B, C and D

5 None of the above

[3 marks]

1 10 Which one of the following statements is true for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

1 $\text{rank}A = 3$ and $\text{nullity}A = 0$

2 $\text{rank}A = 2$ and $\text{nullity}A = 1$

3 $\text{rank}A = 1$ and $\text{nullity}A = 2$

4 $\text{rank}A = 2$ and $\text{nullity}A = 0$

5 None of the above

[3 marks]

[Total for Question 1 $10 \times 3 = 30$ marks]

Answer Questions 2-4 in the answer book

Question 2.

Let $B = \{(1,-1,1),(-1,1,1)\}$ and $C = \{(1,-1,0),(0,0,1)\}$ be subsets of $\mathbb{R}^3$

(a) Show that both the sets B and C are linearly independent sets of vectors with $\text{span}B = \text{span}C$

[12 marks]

(b) Assuming the usual left to right ordering, find the transition matrix $P_{B\to C}$

[2 marks]

(c) Given a basis D of $\mathbb{R}^2$, find the transition matrix $P_{B\to D}$ given

$$P_{C\to D} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

[3 marks]

(d) Use the transition matrix $P_{C\to D}$ in (c) to find D

[8 marks]

[Total for Question 2 25 marks]

Question 3.
Let

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix},$$

and for $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} A \mathbf{y} = 2x_1y_1 + 3x_1y_2 + 3x_2y_1 + 5x_2y_2 \tag{1}$$

(a) Show that the assignment $(\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$ defined in (1) is an inner product

[10 marks]

(b) If $\mathbf{a} = (1, -1)$ and $\mathbf{b} = (1, 1)$, then show that the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent but they are not orthogonal with respect to the inner product in (1)

[3 marks]

(c) Given the vectors $\mathbf{a}$ and $\mathbf{b}$ in (b), the set $\{\mathbf{a}, \mathbf{b}\}$ is hence a basis for $\mathbb{R}^2$
Obtain an orthonormal basis for $\mathbb{R}^2$ from $\{\mathbf{a}, \mathbf{b}\}$ by means of the Gram-Schmidt orthonormalisation process with respect to the inner product defined in (1)

[12 marks]

[Total for Question 3 25 marks]

Question 4.
Let T $M_{22} \to P_2$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + \frac{b-c}{2}x + dx^2,$$

where $a, b, c, d \in \mathbb{R}$

(a) Show that T $M_{22} \to P_2$ is a linear transformation

[4 marks]

(b) Find the matrix representation $[T]_{B',B}$ of T relative to the basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22} and the basis

$$B' = \{1 + x, 1 - x, x^2\}$$

in P_2, ordered from left to right

[12 marks]

INFORMATION SHEET

Vector spaces

Definition . Vector space.

A *vector space* is a non-empty set V with vector addition $+ \quad V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ obeying the axioms

VS1 $\mathbf{u}+\mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$,

VS2 $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$,

VS3 $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

VS4 there exists $\mathbf{0} \in V$ such that $\mathbf{u}+\mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$,

VS5 for all $\mathbf{u} \in V$ there exists $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$,

VS6 $a \quad \mathbf{u} \in V$ for all $a \in \mathbb{R}$, $\mathbf{u} \in V$,

VS7 $a \quad (\mathbf{u}+\mathbf{v}) = a \quad \mathbf{u} + a \quad \mathbf{v}$ for all $a \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in V$,

VS8 $(a+b) \quad \mathbf{u} = a \quad \mathbf{u} + b \quad \mathbf{u}$ for all $a, b \in \mathbb{R}$, $\mathbf{u} \in V$,

VS9 $a \quad (b \quad \mathbf{u}) = (ab) \quad \mathbf{u}$ for all $a, b \in \mathbb{R}$, $\mathbf{u} \in V$,

VS10 $1 \quad \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$

Theorem . VZ.
$\mathbf{0} = 0 \quad \mathbf{u} = a \quad \mathbf{0}$ for all $a \in \mathbb{R}$ and $\mathbf{u} \in V$ in a vector space V

Theorem . VN.
$(-1) \quad \mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in V$ in a vector space V

Definition Subspace.

A subset $W \subseteq V$ of a vector space V is a *subspace* of V if W, with the same vector addition and scalar multiplication as V, is a vector space

Theorem . SS.

A subset $W \subseteq V$ of a vector space V is a *subspace* of V, with the same vector addition + and scalar multiplication as V, if and only if

1 W is not empty

[TURN OVER]

1 $\mathbf{u}+\mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in W$,

1 $a \quad \mathbf{u} \in W$ for all $a \in \mathbb{R}$, $\mathbf{u} \in V$

Definition . Linear independence.

A subset $\{\mathbf{b}_1, \quad , \mathbf{b}_n\} \subseteq V$ in a vector space V is *linearly independent* if and only if

$$c_1 \quad \mathbf{b}_1 + \quad + c_n \quad \mathbf{b}_n = \mathbf{0} \iff c_1 = \quad = c_n = 0$$

Definition . Span.

The *span* of a subset $\{\mathbf{b}_1, \quad , \mathbf{b}_n\} \subseteq V$ in a vector space V is the subspace of V given by

$$\text{span}\{\mathbf{b}_1, \quad , \mathbf{b}_n\} = \{c_1 \quad \mathbf{b}_1 + \quad + c_n \quad \mathbf{b}_n \quad c_1, \quad , c_n \in \mathbb{R}\}$$

Definition . Basis, dimension.

A subset $\{\mathbf{b}_1, \quad , \mathbf{b}_n\} \subseteq V$ in a vector space V is a *basis* for V if and only if

1 $\{\mathbf{b}_1, \quad , \mathbf{b}_n\}$ is linearly independent,

2 $\text{span}\{\mathbf{b}_1, \quad , \mathbf{b}_n\} = V$

If $\{\mathbf{b}_1, \quad , \mathbf{b}_n\} \subseteq V$ is a basis for V then the dimension of V is n, $\dim(V) = n$

Definition . Coordinate matrix.

Let $B = \{\mathbf{b}_1, \quad , \mathbf{b}_n\}$ be a basis for V and let $\mathbf{v} \in V$ Then there exists unique $c_1, \quad , c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1 \quad \mathbf{b}_1 + \quad + c_n \quad \mathbf{b}_n$ The column vector

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \\ c_n \end{bmatrix}$$

is the *coordinate matrix* of $\mathbf{v}$ relative to B

Definition . Transition matrix, change of coordinate matrix.

Let $B_1 = \{\mathbf{b}_1, \quad , \mathbf{b}_n\}$ be a basis for the vector space V, and B_2 be another basis for V The *transition matrix (change of coordinate matrix)* $P_{B_1 \to B_2}$ from B_1 to B_2 is given by

$$P_{B_1 \to B_2} = \left[[\mathbf{b}_1]_{B_2} \qquad [\mathbf{b}_n]_{B_2} \right]$$

Examples . Vector spaces.

- $\mathbb{R}^n$
- The vector space $P_n = \{c_0 + c_1 x + \quad + c_n x^n \quad c_0, \quad , c_n \in \mathbb{R}\}$ of polynomials of degree n or less
- The vector space M_{mn} of $m \times n$ matrices

Inner products

Definition . Inner product.

An *inner product* is a function $\langle \, , \, \rangle \quad V \times V \to \mathbb{R}$ on a vector space V which obeys the axioms

IP1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$,

IP2 $\langle k \quad \mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ for all $k \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in V$,

IP3 $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

IP4 a) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, for all $\mathbf{u} \in V$,

a) $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$

Definition . Orthogonality.

Let $\langle \, , \, \rangle$ denote an inner product on a vector space V If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* to each other

Definition . Unit vector, normalized.

Let $\langle\ ,\ \rangle$ denote an inner product on a vector space V If $\langle \mathbf{u}, \mathbf{u}\rangle = 1$, then $\mathbf{u}$ is a *unit vector (normalized)*

Theorem . Cauchy-Schwarz inequality.

Let $\langle\ ,\ \rangle$ denote an inner product on a vector space V Then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle}$$

for all $\mathbf{u}, \mathbf{v} \in V$

Definition . Gram-Schmidt process.

Let $\langle\ ,\ \rangle$ denote an inner product on a vector space V and let $\{\mathbf{u}_1, \quad , \mathbf{u}_m\}$ be a linearly independent set in V The *Gram-Schmidt process* yields an *orthogonal basis* $\{\mathbf{v}_1, \quad , \mathbf{v}_m\}$ for span$\{\mathbf{u}_1, \quad , \mathbf{u}_m\}$ as follows

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1, \\
\\
\mathbf{v}_m &= \mathbf{u}_m - \frac{\langle \mathbf{u}_m, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \quad - \frac{\langle \mathbf{u}_m, \mathbf{v}_{m-1}\rangle}{\langle \mathbf{v}_{m-1}, \mathbf{v}_{m-1}\rangle}\mathbf{v}_{m-1}
\end{aligned}$$

An *orthonormal basis* $\{\mathbf{v}'_1, \quad , \mathbf{v}'_m\}$ is obtained by setting $\mathbf{v}'_j = \dfrac{\mathbf{v}_j}{\langle \mathbf{v}_j, \mathbf{v}_j\rangle}$

Linear transformations

Definition . Linear transformation.

A function $T \quad V \to W$ between vector spaces V and W is a *linear transformation* if and only if

1 $T(k \quad \mathbf{u}) = k \quad T(\mathbf{u})$ for all $k \in \mathbb{R}$, $\mathbf{u} \in V$

2 $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$

Examples . Linear transformations.

- The trace operation on M_{nn} is a linear transformation tr $\quad M_{nn} \to \mathbb{R}$

- The transpose operation on M_{mn} is a linear transformation

Definition . Kernel, nullity.

The *kernel* of a linear transformation $T : V \to W$ between vector spaces V and W is the subspace

$$\ker(T) = \{\mathbf{v} \in V : T(v) = \mathbf{0}_W\}$$

of V, where $\mathbf{0}_W$ is the zero vector in W. The *nullity* of T is the dimension of $\ker(T)$

Definition . Range, rank.

The *range* of a linear transformation $T : V \to W$ between vector spaces V and W is the subspace

$$R(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

of W. The *rank* of T is the dimension of $R(T)$

Definition . One-to-one, injective, inverse.

A linear transformation $T : V \to W$ between vector spaces V and W is *one-to-one* if and only if

$$T(\mathbf{u}) = T(\mathbf{v}) \iff \mathbf{u} = \mathbf{v}$$

A one-to-one linear transformation $T : V \to W$ has an inverse linear transformation $T^{-1} : R(T) \to V$ satisfying $T^{-1}(T(\mathbf{u})) = \mathbf{u}$ for all $\mathbf{u} \in V$

Definition . Onto, surjective.

A linear transformation $T : V \to W$ between vector spaces V and W is *onto* if and only if $R(T) = W$

Theorem . TO.
If V and W are finite dimensional vector spaces and $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$. If $\dim(V) = \dim(W)$, then T is onto if and only if T is one-to-one

Definition . Isomorphism, bijection.

A one-to-one and onto linear transformation $T : V \to W$ between vector spaces V and W is an *isomorphism (bijection)*. If an isomorphism between V and W exists, then V and W are *isomorphic*

Theorem . VI.
Every vector space V with $\dim(V) = n$ is isomorphic to $\mathbb{R}^n$

Definition . Matrix representation of a linear transformation.

Let $B_V = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V, and B_W be a basis for the vector space W The *matrix representation* $[T]_{B_W,B_V}$ of a linear transformation $T : V \to W$ is given by

$$[T]_{B_W,B_V} = \left[\, [T(\mathbf{b}_1)]_{B_W} \quad \cdots \quad [T(\mathbf{b}_n)]_{B_W} \,\right]$$

When $V = W$ and $B_V = B_W$, we write $[T]_{B_V} = [T]_{B_V,B_V}$

Matrices

Definition . Column space, row space, rank.

Let A be an $m \times n$ matrix with columns $A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix}$ and rows $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$

The *column space* of A is $\text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ and the *row space* of A is $\text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ The *rank* of A is the dimension of the column and row spaces, $\text{rank}(A) = \dim(\text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}) = \dim(\text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\})$

Definition . Null space, nullity.

The *null space* of an $m \times n$ matrix A is the subspace

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

The *nullity* of T is the dimension of $N(A)$

Theorem . RN.
$\text{rank}(A) + \text{nullity}(A) = n$ for every $m \times n$ matrix A

Definition . Eigenvalue, eigenvector.

Let A be an $n \times n$ matrix If $A\mathbf{x} = \lambda\mathbf{x}$, for $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$, then λ is an *eigenvalue* of A and $\mathbf{x}$ is an *eigenvector* of A corresponding to the eigenvalue λ

Definition . Eigenspace, geometric multiplicity.

Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A Then

$$E_\lambda = \{\mathbf{x} \in \mathbb{C}^n \quad A\mathbf{x} = \lambda\mathbf{x}\}$$

is a vector space, called the *eigenspace* for the eigenvalue λ of A The *geometric multiplicity* of λ is $\dim(E_\lambda)$

Definition . Characteristic equation, characteristic polynomial.

Let A be an $n \times n$ matrix Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ satisfies the *characteristic equation* $\det(\lambda I_n - A) = 0$, where I_n is the $n \times n$ identity matrix The polynomial $\det(xI_n - A)$ is the *characteristic polynomial* in the variable x

Definition . Algebraic multiplicity.

Let A be an $n \times n$ matrix with eigenvalue λ The *algebraic multiplicity* of λ is the largest number $a \in \mathbb{N}$ such that $(x - \lambda)^a$ is a factor of the characteristic polynomial $\det(xI_n - A)$

Definition . Diagonalizable.

An $n \times n$ matrix A is *diagonalizable* if and only if A is similar to some $n \times n$ diagonal matrix D, i e $A = PDP^{-1}$ for some $n \times n$ diagonal matrix D and non-singular $n \times n$ matrix P

Theorem . DI.
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors

Theorem . DD.
If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable

Theorem . DS.
If an $n \times n$ matrix A is symmetric, then A is diagonalizable

Theorem . DM.
For a square matrix A, the algebraic and geometric multiplicity are equal for each eigenvalue of A if and only if A is diagonalizable

Definition . Trace.

The *trace* of a square matrix is the sum of its diagonal entries

$$\mathrm{tr}\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = a_{11} + a_{22} + \qquad + a_{nn}$$

Theorem . CT
For all $n \times n$ matrices A, B and C we have $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$ Consequently $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

Definition . Transpose.

The transpose of a matrix is obtained by interchanging corresponding rows and columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ & & & \cdot \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}$$

Theorem . TT.
For all $m \times n$ matrices A we have $(A^T)^T = A$

Theorem . TI.
For all $n \times n$ matrices A we have $\operatorname{tr}(A) = \operatorname{tr}(A^T)$

Determinants

For 2×2 and 3×3 matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Cofactor expansion along the j-th row.

$$\begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdot & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & & a_{1,k-1} & a_{1,k+1} & \cdot\cdot & a_{1n} \\ a_{21} & a_{22} & & a_{2,k-1} & a_{2,k+1} & & a_{2n} \\ & & \cdot & \cdot & & & \\ a_{j-1,1} & a_{j-1,2} & & a_{j-1,k-1} & a_{j-1,k+1} & & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & & a_{j+1,k-1} & a_{j+1,k+1} & \cdot & a_{j+1,n} \\ \cdot & & & & & & \\ a_{n1} & a_{n2} & & a_{n,k-1} & a_{n,k+1} & & a_{nn} \end{vmatrix}$$

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EXAMINATION MARK READING SHEET — EKSAMEN-MERKLEESBLAD

PART 1 (GENERAL/ALGEMEEN) DEEL 1

STUDY UNIT e g PSY100 X / STUDIE EENHEID bv PSY100-X — **1**

PAPER NUMBER / VRAESTELNOMMER — **2**

INITIALS AND SURNAME / VOORLETTERS EN VAN — **3**

DATE OF EXAMINATION / DATUM VAN EKSAMEN — **4**

EXAMINATION CENTRE (E G PRETORIA) / EKSAMENSENTRUM (BV PRETORIA) — **5**

STUDENT NUMBER / STUDENTENOMMER — **6**

[0] [0] [0] [0] [0] [0] [0] [0]
[1] [1] [1] [1] [1] [1] [1] [1]
[2] [2] [2] [2] [2] [2] [2] [2]
[3] [3] [3] [3] [3] [3] [3] [3]
[4] [4] [4] [4] [4] [4] [4] [4] — **7**
[5] [5] [5] [5] [5] [5] [5] [5]
[6] [6] [6] [6] [6] [6] [6] [6]
[7] [7] [7] [7] [7] [7] [7] [7]
[8] [8] [8] [8] [8] [8] [8] [8]
[9] [9] [9] [9] [9] [9] [9] [9]

UNIQUE PAPER NO / UNIEKE VRAESTEL NR — **8**

[0] [0] [0] [0] [0] [0]
[1] [1] [1] [1] [1] [1]
[2] [2] [2] [2] [2] [2]
[3] [3] [3] [3] [3] [3]
[4] [4] [4] [4] [4] [4] — **9**
[5] [5] [5] [5] [5] [5]
[6] [6] [6] [6] [6] [6]
[7] [7] [7] [7] [7] [7]
[8] [8] [8] [8] [8] [8]
[9] [9] [9] [9] [9] [9]

For use by examination invigilator
Vir gebruik deur eksamenopsiener

IMPORTANT

1. USE ONLY AN HB PENCIL TO COMPLETE THIS SHEET
2. MARK LIKE THIS
3. CHECK THAT YOUR INITIALS AND SURNAME HAS BEEN FILLED IN CORRECTLY
4. ENTER YOUR STUDENT NUMBER FROM LEFT TO RIGHT
5. CHECK THAT YOUR STUDENT NUMBER HAS BEEN FILLED IN CORRECTLY
6. CHECK THAT THE UNIQUE NUMBER HAS BEEN FILLED IN CORRECTLY
7. CHECK THAT ONLY ONE ANSWER PER QUESTION HAS BEEN MARKED
8. DO NOT FOLD

BELANGRIK

1. GEBRUIK SLEGS 'N HB POTLOOD OM HIERDIE BLAD TE VOLTOOI
2. MERK AS VOLG
3. KONTROLEER DAT U VOORLETTERS EN VAN REG INGEVUL IS
4. VUL U STUDENTENOMMER VAN LINKS NA REGS IN
5. KONTROLEER DAT U DIE KORREKTE STUDENTENOMMER VERSTREK HET
6. KONTROLEER DAT DIE UNIEKE NOMMER REG INGEVUL IS
7. MAAK SEKER DAT NET EEN ALTERNATIEF PER VRAAG GEMERK IS
8. MOENIE VOU NIE

PART 2 (ANSWERS/ANTWOORDE) DEEL 2

10

1	[1] [2] [3] [4] [5]	36	[1] [2] [3] [4] [5]	71	[1] [2] [3] [4] [5]	106	[1] [2] [3] [4] [5]
2	[1] [2] [3] [4] [5]	37	[1] [2] [3] [4] [5]	72	[1] [2] [3] [4] [5]	107	[1] [2] [3] [4] [5]
3	[1] [2] [3] [4] [5]	38	[1] [2] [3] [4] [5]	73	[1] [2] [3] [4] [5]	108	[1] [2] [3] [4] [5]
4	[1] [2] [3] [4] [5]	39	[1] [2] [3] [4] [5]	74	[1] [2] [3] [4] [5]	109	[1] [2] [3] [4] [5]
5	[1] [2] [3] [4] [5]	40	[1] [2] [3] [4] [5]	75	[1] [2] [3] [4] [5]	110	[1] [2] [3] [4] [5]
6	[1] [2] [3] [4] [5]	41	[1] [2] [3] [4] [5]	76	[1] [2] [3] [4] [5]	111	[1] [2] [3] [4] [5]
7	[1] [2] [3] [4] [5]	42	[1] [2] [3] [4] [5]	77	[1] [2] [3] [4] [5]	112	[1] [2] [3] [4] [5]
8	[1] [2] [3] [4] [5]	43	[1] [2] [3] [4] [5]	78	[1] [2] [3] [4] [5]	113	[1] [2] [3] [4] [5]
9	[1] [2] [3] [4] [5]	44	[1] [2] [3] [4] [5]	79	[1] [2] [3] [4] [5]	114	[1] [2] [3] [4] [5]
10	[1] [2] [3] [4] [5]	45	[1] [2] [3] [4] [5]	80	[1] [2] [3] [4] [5]	115	[1] [2] [3] [4] [5]
11	[1] [2] [3] [4] [5]	46	[1] [2] [3] [4] [5]	81	[1] [2] [3] [4] [5]	116	[1] [2] [3] [4] [5]
12	[1] [2] [3] [4] [5]	47	[1] [2] [3] [4] [5]	82	[1] [2] [3] [4] [5]	117	[1] [2] [3] [4] [5]
13	[1] [2] [3] [4] [5]	48	[1] [2] [3] [4] [5]	83	[1] [2] [3] [4] [5]	118	[1] [2] [3] [4] [5]
14	[1] [2] [3] [4] [5]	49	[1] [2] [3] [4] [5]	84	[1] [2] [3] [4] [5]	119	[1] [2] [3] [4] [5]
15	[1] [2] [3] [4] [5]	50	[1] [2] [3] [4] [5]	85	[1] [2] [3] [4] [5]	120	[1] [2] [3] [4] [5]
16	[1] [2] [3] [4] [5]	51	[1] [2] [3] [4] [5]	86	[1] [2] [3] [4] [5]	121	[1] [2] [3] [4] [5]
17	[1] [2] [3] [4] [5]	52	[1] [2] [3] [4] [5]	87	[1] [2] [3] [4] [5]	122	[1] [2] [3] [4] [5]
18	[1] [2] [3] [4] [5]	53	[1] [2] [3] [4] [5]	88	[1] [2] [3] [4] [5]	123	[1] [2] [3] [4] [5]
19	[1] [2] [3] [4] [5]	54	[1] [2] [3] [4] [5]	89	[1] [2] [3] [4] [5]	124	[1] [2] [3] [4] [5]
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21	[1] [2] [3] [4] [5]	56	[1] [2] [3] [4] [5]	91	[1] [2] [3] [4] [5]	126	[1] [2] [3] [4] [5]
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23	[1] [2] [3] [4] [5]	58	[1] [2] [3] [4] [5]	93	[1] [2] [3] [4] [5]	128	[1] [2] [3] [4] [5]
24	[1] [2] [3] [4] [5]	59	[1] [2] [3] [4] [5]	94	[1] [2] [3] [4] [5]	129	[1] [2] [3] [4] [5]
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26	[1] [2] [3] [4] [5]	61	[1] [2] [3] [4] [5]	96	[1] [2] [3] [4] [5]	131	[1] [2] [3] [4] [5]
27	[1] [2] [3] [4] [5]	62	[1] [2] [3] [4] [5]	97	[1] [2] [3] [4] [5]	132	[1] [2] [3] [4] [5]
28	[1] [2] [3] [4] [5]	63	[1] [2] [3] [4] [5]	98	[1] [2] [3] [4] [5]	133	[1] [2] [3] [4] [5]
29	[1] [2] [3] [4] [5]	64	[1] [2] [3] [4] [5]	99	[1] [2] [3] [4] [5]	134	[1] [2] [3] [4] [5]
30	[1] [2] [3] [4] [5]	65	[1] [2] [3] [4] [5]	100	[1] [2] [3] [4] [5]	135	[1] [2] [3] [4] [5]
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32	[1] [2] [3] [4] [5]	67	[1] [2] [3] [4] [5]	102	[1] [2] [3] [4] [5]	137	[1] [2] [3] [4] [5]
33	[1] [2] [3] [4] [5]	68	[1] [2] [3] [4] [5]	103	[1] [2] [3] [4] [5]	138	[1] [2] [3] [4] [5]
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35	[1] [2] [3] [4] [5]	70	[1] [2] [3] [4] [5]	105	[1] [2] [3] [4] [5]	140	[1] [2] [3] [4] [5]

Specimen only

MARK READING SHEET INSTRUCTIONS

Your mark reading sheet is marked by computer and should therefore be filled in thoroughly and correctly

USE ONLY AN HB PENCIL TO COMPLETE YOUR MARK READING SHEET

PLEASE DO NOT FOLD OR DAMAGE YOUR MARK READING SHEET

Consult the illustration of a mark reading sheet on the reverse of this page and follow the instructions step by step when working on your sheet

Instruction numbers ❶ to ❿ refer to spaces on your mark reading sheet which you should fill in as follows

❶ Write your paper code in these eight squares, for instance

P	S	Y	1	0	0	-	X

❷ The paper number pertains only to first-level courses consisting of two papers

WRITE [0][1] for the first paper and [0][2] for the second If only one paper, then leave blank

❸ Fill in your initials and surname

❹ Fill in the date of the examination

❺ Fill in the name of the examination centre

❻ WRITE the digits of your student number HORIZONTALLY (from left to right) Begin by filling in the first digit of your student number in the first square on the left, then fill in the other digits, each one in a separate square

❼ In each vertical column mark the digit that corresponds to the digit in your student number as follows [-]

❽ WRITE your unique paper number HORIZONTALLY
NB Your unique paper number appears at the top of your examination paper and consists only of digits (e g 403326)

❾ In each vertical column mark the digit that corresponds to the digit number in your unique paper number as follows [-]

❿ Question numbers 1 to 140 indicate corresponding question numbers in your examination paper The five spaces with digits 1 to 5 next to each question number indicate an alternative answer to each question The spaces of which the number correspond to the answer you have chosen for each question and should be marked as follows [-]

♦ For official use by the invigilator Do not fill in any information here