UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611

May/June 2018

LINEAR ALGEBRA

Duration 2 Hours

140 Marks

EXAMINERS

FIRST DR PP GHOSH

SECOND DR ZE MPONO

Closed book examination.

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

[TURN OVER]

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This paper consists of 14 pages, including the first cover page

Instructions

- The notation is the same as in the Assignments/Study Guide
- The questions for this examination appear on pages 2 - 6
- A full list of definitions and theorems appear on **INFORMATION SHEET** on pages 7 - 14
- Answer as much as you can If your score on the paper is S then the mark M for this examination shall be

$$M = \begin{cases} S, & \text{if } 0 \leq S \leq 100 \\ 100 & \text{if } S \geq 100 \end{cases} = \min\{S, 100\}$$

[TURN OVER]

Question 1. (a) Consider the linear space P_4 of all polynomials of degree at most 4 with its usual operations

Show that the subset P_2 of all polynomials of degree at most 2 is a linear subspace of P_4

[4 marks]

(b) Explain with reasons the truth/falsity of the statement

The set $S = \{(x,y) \in \mathbb{R}^2 \quad y = 2x\}$ is a linear subspace of $\mathbb{R}^2$ but the set $T = \{(x,y) \in \mathbb{R}^2 \quad y = x^2\}$ is **<u>not</u>** a linear subspace of $\mathbb{R}^2$

[4 marks]

(c) Consider the linear space M_{23} of all 2×3 matrices with real entries

Show that the subset

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \right\}$$

is a linearly independent subset of M_{23}

Is the subset

$$B' = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \right\}$$

a linearly independent set too? Explain your answer

[4 marks]

(d) Show that the subset $B = \{1, 1+x, 1+x+x^2\}$, ordered from left to right, is a basis for the linear space P_2

[4 marks]

(e) Given the basis $B = \{1, 1+x, 1+x+x^2\}$ (ordered from left to right) obtain the coordinates of the polynomial $p(x) = 2x^2 - 5x + 6$ with respect to the basis B

[2 marks]

(f) Explain without any computation whether the subset $C = \{1, x, 1-x, 1+x+x^2\}$ of P_2 is a linearly independent set or a linearly dependent set

[3 marks]

[TURN OVER]

(g) Show that the null space of the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{pmatrix}$ is spanned by the vector $(0, 1, 1)$

Hence or otherwise find the dimension of the subspace

$$R = \left\{Ax \quad x \in \mathbb{R}^3\right\} = \left\{(x + y - z, y - z, x, x - 2y + 2z) \quad x, y, z \in \mathbb{R}\right\} \subseteq \mathbb{R}^4$$

of $\mathbb{R}^4$

[4 marks]

(h) Show that the matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$ is **not** diagonalisable

Hint Show that 0 is an eigenvalue of multiplicity > 1

[5 marks]

[Total Marks for 1 30 marks]

Question 2. Let $T \quad P_3 \to M_{22}$ be defined by

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a - c & a - d \\ b - d & b - c \end{pmatrix}$$

For instance $T(1 + 4x + 2x^2 - 3x^3) = \begin{pmatrix} -1 & 4 \\ 7 & 2 \end{pmatrix}$

Let $B = \left\{1, x, x^2, x^3\right\}$ be the standard basis of P_3 (ordered from left to right) and consider the given basis $B' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ (ordered from left to right) for M_{22}

Show that T is a linear transformation and find the matrix representation $\left[T\right]_{B,B'}$ of the linear transformation with respect to the bases B for P_3 and B' for M_{22}

[Total Marks for Question 2 15 marks]

Question 3. Consider the basis $B = \{(0,1),(1,0)\}$ (ordered from left to right) of $\mathbb{R}^2$

Find the basis D of $\mathbb{R}^2$ if $[P]_{B\to D} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$

[Total Marks for Question 3 5 marks]

Question 4. Consider the linear space $\mathbb{R}^3$ and let $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ Define

$$\left\langle \mathbf{x}, \mathbf{y} \right\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 \tag{1}$$

Let $B = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

(a) Show that $\left\langle \mathbf{x}, \mathbf{y} \right\rangle$ in equation (1) is an inner product on $\mathbb{R}^3$

[10 marks]

(b) Show that B is a linearly independent set of vectors in $\mathbb{R}^3$ but is **not** an orthogonal set with respect to the inner product defined in equation (1)

[4 marks]

(c) Use the Gram-Schmidt orthogonalisation process to B to obtain an orthonormal basis with respect to the inner product defined in equation (1)

[16 marks]

[Total Marks for Question 4 30 marks]

Question 5. (a) Let C be a 3×3 matrix and P be an invertible 3×3 matrix whose columns are the vectors $\mathbf{p}$ $\mathbf{q}$ and $\mathbf{r}$

(i) Show that the matrix CP has columns $C\mathbf{p}$, $C\mathbf{q}$ and $C\mathbf{r}$

[5 marks]

(ii) Show that for any diagonal matrix $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, the matrix PD has columns $a\mathbf{p}$ $b\mathbf{q}$ and $c\mathbf{r}$

[3 marks]

[TURN OVER]

(iii) Show that the matrix $P^{-1}CP$ is a diagonal matrix, if and only if, there exist scalars a b and c such that a, b, c are eigenvalues of A $\mathbf{p}$ is an eigenvector of A corresponding to a, $\mathbf{q}$ is an eigenvector of A corresponding to b and $\mathbf{r}$ is an eigenvector of A corresponding to c

[8 marks]

(b) Show that if B is an invertible matrix, λ an eigenvalue of B and $\mathbf{v}$ an eigenvector of B corresponding to the eigenvalue λ then λ^{-1} is an eigenvalue of A^{-1} and $\mathbf{v}$ is an eigenvector of A^{-1} corresponding to the eigenvalue λ^{-1}

[6 marks]

(c) Let A be a square matrix whose characteristic polynomial is $p(x) = x^3 - x$

(i) What is the size of the matrix A? (Recall that the size of a matrix A with m rows and n columns is $m \times n$)

[2 marks]

(ii) Find the eigenvalues of A and their algebraic and geometric multiplicities

[6 marks]

(iii) Show that the matrix A is diagonalisable

Hint. You might consider eigenvectors for the eigenvalues and then use the results in the parts (a) (i), (a) (ii) & (a) (iii)

[8 marks]

(iv) Hence, or otherwise, obtain the characteristic polynomial of the matrix A^2

[5 marks]

(v) What can you say about the algebraic and geometric multiplicities of the eigenvalues of A^2?

Hint: You could use the result in part (a) (iii)

[15 marks]

(vi) Is the matrix A invertible?

Hint. You could use the result in part (b)

[2 marks]

[Total Marks for Question 5: 60 marks]

INFORMATION SHEET

Vector spaces

Definition . Vector space.
A *vector space* is a non-empty set V with vector addition $+ \quad V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ obeying the axioms

VS1 $\mathbf{u} + \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$

VS2 $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$,

VS3 $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

VS4 there exists $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$,

VS5 for all $\mathbf{u} \in V$ there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$,

VS6 $a \quad \mathbf{u} \in V$ for all $a \in \mathbb{R}$, $\mathbf{u} \in V$,

VS7 $a \quad (\mathbf{u} + \mathbf{v}) = a \quad \mathbf{u} + a \quad \mathbf{v}$ for all $a \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in V$,

VS8 $(a + b) \quad \mathbf{u} = a \quad \mathbf{u} + b \quad \mathbf{u}$ for all $a, b \in \mathbb{R}$, $\mathbf{u} \in V$,

VS9 $a \quad (b \quad \mathbf{u}) = (ab) \quad \mathbf{u}$ for all $a, b \in \mathbb{R}$, $\mathbf{u} \in V$,

VS10 $1 \quad \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$

Theorem . VZ. $\mathbf{0} = 0 \quad \mathbf{u} = a \quad \mathbf{0}$ for all $a \in \mathbb{R}$ and $\mathbf{u} \in V$ in a vector space V
Theorem . VN. $(-1) \quad \mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in V$ in a vector space V
Definition . Subspace
A subset $W \subseteq V$ of a vector space V is a *subspace* of V if W, with the same vector addition and scalar multiplication as V, is a vector space
Theorem . SS.
A subset $W \subseteq V$ of a vector space V is a *subspace* of V, with the same vector addition $+$ and scalar multiplication as V, if and only if

1 W is not empty,

1 $\mathbf{u} + \mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in W$,

1 $a \quad \mathbf{u} \in W$ for all $a \in \mathbb{R}$, $\mathbf{u} \in V$

Definition . Linear independence
A subset $\{\mathbf{b}_1, \quad , \mathbf{b}_n\} \subseteq V$ in a vector space V is *linearly independent* if and only if

$$c_1 \quad \mathbf{b}_1 + \quad + c_n \quad \mathbf{b}_n = \mathbf{0} \iff c_1 = \quad = c_n = 0$$

Definition Span.
The *span* of a subset $\{\mathbf{b}_1, \quad, \mathbf{b}_n\} \subseteq V$ in a vector space V is the subspace of V given by

$$\text{span}\{\mathbf{b}_1, \quad, \mathbf{b}_n\} = \{c_1 \quad \mathbf{b}_1 + \quad + c_n \quad \mathbf{b}_n \quad c_1, \quad, c_n \in \mathbb{R}\}$$

Definition Basis, dimension.
A subset $\{\mathbf{b}_1, \quad, \mathbf{b}_n\} \subseteq V$ in a vector space V is a *basis* for V if and only if

1 $\{\mathbf{b}_1, \quad, \mathbf{b}_n\}$ is linearly independent,

2 $\text{span}\{\mathbf{b}_1, \quad, \mathbf{b}_n\} = V$

If $\{\mathbf{b}_1, \quad, \mathbf{b}_n\} \subseteq V$ is a basis for V then the dimension of V is n, $\dim(V) = n$
Definition . Coordinate matrix.
Let $B = \{\mathbf{b}_1, \quad, \mathbf{b}_n\}$ be a basis for V and let $\mathbf{v} \in V$ Then there exists unique $c_1, \quad, c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1 \quad \mathbf{b}_1 + \quad + c_n \quad \mathbf{b}_n$ The column vector

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \\ c_n \end{bmatrix}$$

is the *coordinate matrix* of $\mathbf{v}$ relative to B

Definition . Transition matrix, change of coordinate matrix
Let $B_1 = \{\mathbf{b}_1, \quad , \mathbf{b}_n\}$ be a basis for the vector space V, and B_2 be another basis for V The *transition matrix (change of coordinate matrix)* $P_{B_1 \to B_2}$ from B_1 to B_2 is given by

$$P_{B_1 \to B_2} = \left[[\mathbf{b}_1]_{B_2} \qquad [\mathbf{b}_n]_{B_2} \right]$$

Examples . Vector spaces.

- $\mathbb{R}^n$
- The vector space $P_n = \{c_0 + c_1 x + \quad + c_n x^n \quad c_0, \quad , c_n \in \mathbb{R}\}$ of polynomials of degree n or less
- The vector space M_{mn} of $m \times n$ matrices

Inner products

Definition Inner product
An *inner product* is a function $\langle \, , \, \rangle \quad V \times V \to \mathbb{R}$ on a vector space V which obeys the axioms

IP1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$,

IP2 $\langle k \quad \mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$ for all $k \in \mathbb{R}$, $\mathbf{u} \quad \mathbf{v} \in V$

IP3 $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

IP4 a) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, for all $\mathbf{u} \in V$,
a) $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$

Definition Orthogonality.
Let $\langle \, , \, \rangle$ denote an inner product on a vector space V If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* to each other

Definition . Unit vector, normalized
Let $\langle\ ,\ \rangle$ denote an inner product on a vector space V If $\langle \mathbf{u}, \mathbf{u}\rangle = 1$, then $\mathbf{u}$ is a *unit vector (normalized)*
Theorem Cauchy-Schwarz inequality.
Let $\langle\ ,\ \rangle$ denote an inner product on a vector space V Then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle}$$

for all $\mathbf{u}, \mathbf{v} \in V$
Definition . Gram-Schmidt process.
Let $\langle\ ,\ \rangle$ denote an inner product on a vector space V and let $\{\mathbf{u}_1, \ \ , \mathbf{u}_m\}$ be a linearly independent set in V The *Gram-Schmidt process* yields an *orthogonal basis* $\{\mathbf{v}_1, \ \ , \mathbf{v}_m\}$ for $\operatorname{span}\{\mathbf{u}_1, \ \ , \mathbf{u}_m\}$ as follows

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1, \\
\\
\mathbf{v}_m &= \mathbf{u}_m - \frac{\langle \mathbf{u}_m, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \quad - \frac{\langle \mathbf{u}_m, \mathbf{v}_{m-1}\rangle}{\langle \mathbf{v}_{m-1}, \mathbf{v}_{m-1}\rangle}\mathbf{v}_{m-1}
\end{aligned}$$

An *orthonormal basis* $\{\mathbf{v}'_1, \ \ , \mathbf{v}'_m\}$ is obtained by setting $\mathbf{v}'_j = \dfrac{\mathbf{v}_j}{\langle \mathbf{v}_j, \mathbf{v}_j\rangle}$

Linear transformations

Definition Linear transformation.
A function $T \ \ V \to W$ between vector spaces V and W is a *linear transformation* if and only if

1 $T(k \ \ \mathbf{u}) = k \ \ T(\mathbf{u})$ for all $k \in \mathbb{R}$, $\mathbf{u} \in V$

2 $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$

Examples . Linear transformations.

- The trace operation on M_{nn} is a linear transformation tr $\ M_{nn} \to \mathbb{R}$
- The transpose operation on M_{mn} is a linear transformation

Definition Kernel, nullity
The *kernel* of a linear transformation $T \ \ V \to W$ between vector spaces V and W is the subspace

$$\ker(T) = \{\mathbf{v} \in V \quad T(v) = \mathbf{0}_W\}$$

of V, where $\mathbf{0}_W$ is the zero vector in W The *nullity* of T is the dimension of $\ker(T)$
Definition . Range, rank
The *range* of a linear transformation $T \ \ V \to W$ between vector spaces V and W is the subspace

$$R(T) = \{T(\mathbf{v}) \quad \mathbf{v} \in V\}$$

of W The *rank* of T is the dimension of $R(T)$

Definition . One-to-one, injective, inverse
A linear transformation $T : V \to W$ between vector spaces V and W is *one-to-one* if and only if

$$T(\mathbf{u}) = T(\mathbf{v}) \iff \mathbf{u} = \mathbf{v}$$

A one-to-one linear transformation $T : V \to W$ has an inverse linear transformation $T^{-1} : R(T) \to V$ satisfying $T^{-1}(T(\mathbf{u})) = \mathbf{u}$ for all $\mathbf{u} \in V$

Definition . Onto, surjective
A linear transformation $T : V \to W$ between vector spaces V and W is *onto* if and only if $R(T) = W$

Theorem TO If V and W are finite dimensional vector spaces and $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$ If $\dim(V) = \dim(W)$, then T is onto if and only if T is one-to-one

Definition . Isomorphism, bijection.
A one-to-one and onto linear transformation $T : V \to W$ between vector spaces V and W is an *isomorphism (bijection)* If an isomorphism between V and W exists, then V and W are *isomorphic*

Theorem . VI. Every vector space V with $\dim(V) = n$ is isomorphic to $\mathbb{R}^n$
Definition . Matrix representation of a linear transformation.
Let $B_V = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis for the vector space V, and B_W be a basis for the vector space W. The *matrix representation* $[T]_{B_W,B_V}$ of a linear transformation $T: V \to W$ is given by

$$[T]_{B_W,B_V} = \left[[T(\mathbf{b}_1)]_{B_W} \quad \cdots \quad [T(\mathbf{b}_n)]_{B_W} \right]$$

When $V = W$ and $B_V = B_W$, we write $[T]_{B_V} = [T]_{B_V,B_V}$

Matrices

Definition . Column space, row space, rank.

Let A be an $m \times n$ matrix with columns $A = \begin{bmatrix}\mathbf{c}_1 & \cdots & \mathbf{c}_n\end{bmatrix}$ and rows $A = \begin{bmatrix}\mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m\end{bmatrix}$
The *column space* of A is $\text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ and the *row space* of A is $\text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$. The *rank* of A is the dimension of the column and row spaces, $\text{rank}(A) = \dim(\text{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}) = \dim(\text{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\})$
Definition . Null space, nullity.
The *null space* of an $m \times n$ matrix A is the subspace

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

The *nullity* of T is the dimension of $N(A)$
Theorem . RN. $\text{rank}(A) + \text{nullity}(A) = n$ for every $m \times n$ matrix A
Definition . Eigenvalue, eigenvector.
Let A be an $n \times n$ matrix. If $A\mathbf{x} = \lambda\mathbf{x}$, for $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$, then λ is an *eigenvalue* of A and $\mathbf{x}$ is an *eigenvector* of A corresponding to the eigenvalue λ

Definition . Eigenspace, geometric multiplicity.
Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. Then

$$E_\lambda = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \lambda\mathbf{x}\}$$

is a vector space, called the *eigenspace* for the eigenvalue λ of A. The *geometric multiplicity* of λ is $\dim(E_\lambda)$.

Definition . Characteristic equation, characteristic polynomial
Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ satisfies the *characteristic equation* $\det(\lambda I_n - A) = 0$, where I_n is the $n \times n$ identity matrix. The polynomial $\det(xI_n - A)$ is the *characteristic polynomial* in the variable x.

Definition . Algebraic multiplicity
Let A be an $n \times n$ matrix with eigenvalue λ. The *algebraic multiplicity* of λ is the largest number $a \in \mathbb{N}$ such that $(x - \lambda)^a$ is a factor of the characteristic polynomial $\det(xI_n - A)$.

Theorem . AG. Given any square matrix A and any eigen value λ of A, the geometric multiplicity $\dim E_\lambda$ of λ is at most as the algebraic multiplicity of λ, i.e., $\dim E_\lambda \leq$ algebraic multiplicity of λ.

Definition . Diagonalizable.
An $n \times n$ matrix A is *diagonalizable* if and only if A is similar to some $n \times n$ diagonal matrix D, i.e. $A = PDP^{-1}$ for some $n \times n$ diagonal matrix D and non-singular $n \times n$ matrix P.

Theorem . DI. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Theorem . DD. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Theorem . DS. If an $n \times n$ matrix A is symmetric, then A is diagonalizable.

Theorem . DM. For a square matrix A, the algebraic and geometric multiplicity are equal for each eigenvalue of A if and only if A is diagonalizable.

Definition . Trace.
The *trace* of a square matrix is the sum of its diagonal entries:

$$\operatorname{tr}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} + a_{22} + \cdots + a_{nn}$$

Theorem . CT. For all $n \times n$ matrices A, B and C we have $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$. Consequently $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Definition . Transpose.
The transpose of a matrix is obtained by interchanging corresponding rows and columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Theorem . TT. For all $m \times n$ matrices A we have $(A^T)^T = A$.

Theorem . TI. For all $n \times n$ matrices A we have $\operatorname{tr}(A) = \operatorname{tr}(A^T)$.

Determinants

For 2×2 and 3×3 matrices

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Cofactor expansion along the j-th row

$$\begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & & a_{1,k-1} & a_{1,k+1} & & a_{1n} \\ a_{21} & a_{22} & & a_{2,k-1} & a_{2,k+1} & & a_{2n} \\ & & & & & & \\ a_{j-1,1} & a_{j-1,2} & & a_{j-1,k-1} & a_{j-1,k+1} & & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & & a_{j+1,k-1} & a_{j+1,k+1} & & a_{j+1,n} \\ & & & & & & \\ a_{n1} & a_{n2} & & a_{n,k-1} & a_{n,k+1} & & a_{nn} \end{vmatrix}$$

Cofactor expansion along the k-th column

$$\begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & & a_{1,k-1} & a_{1,k+1} & & a_{1n} \\ a_{21} & a_{22} & & a_{2,k-1} & a_{2,k+1} & & a_{2n} \\ & & & & & & \\ a_{j-1,1} & a_{j-1,2} & & a_{j-1,k-1} & a_{j-1,k+1} & & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & & a_{j+1,k-1} & a_{j+1,k+1} & & a_{j+1,n} \\ & & & & & & \\ a_{n1} & a_{n2} & & a_{n,k-1} & a_{n,k+1} & & a_{nn} \end{vmatrix}$$

TOTAL. [140]

First examiner Dr Partha Pratim Ghosh
Second examiner Dr Z E Mpono