UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611
SECOND PAPER

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Linear Algebra

Duration 2 Hours

110 Marks

EXAMINERS
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Closed book examination.

Instructions

- The paper consists of 4 pages, including the first cover page
- There are six questions in the paper and the total marks is 110 Answer all the questions

[TURN OVER]

Question 1. Suppose A is a square matrix with characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$

(a) What is the order of the matrix A?

(b) Is A invertible?

(c) Is A diagonalisable?

(d) Find the eigenvalues of A^2

[Marks for Question 1 $1 + 1 + 2 + 1 = 5$ marks]

Question 2. Let $A = \begin{pmatrix} -3 & 1 & 0 \\ -6 & 2 & 0 \\ -3 & 1 & 0 \end{pmatrix}$

(a) Find the characteristic polynomial of A and show that the eigenvalues are 0 and -1

(b) Find a basis for each eigenspace of A

(c) Explain why is A diagonalisable

(d) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$

(e) Hence, or otherwise, calculate A^{2018}

[Marks for Question 2 $4 + 6 + 2 + 3 + 5 = 20$ marks]

Question 3. In each of the following cases explain whether $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}$ is a linear transformation, if it is, supply a proof, if not, supply a counterexample

(a) $T(a, b) = a + b$

(b) $T(a, b) = ab$

(c) $T(a, b) = |a|^b$

(d) $T(a, b) = a - b$

[Marks for Question 3 $2\frac{1}{2} \times 4 = 10$ marks]

Question 4 Let $\mathbb{R}[x]_{\deg\leq 2}$ be the vector space of all polynomial functions in a single variable x with real coefficients and degree at most 2

Let $\mathbb{R}[x]_{\deg\leq 2} \xrightarrow{T} \mathbb{R}[x]_{\deg\leq 2}$ be defined by $T(p(x)) = p(x-1)$

Let $U = \langle 1, x, x^2 \rangle$ be the usual basis for $\mathbb{R}[x]_{\deg\leq 2}$ and $B = \langle 1+x+x^2, 2x+x^2, x+x^2 \rangle$ be another basis for $\mathbb{R}[x]_{\deg\leq 2}$

Let, for any polynomial $p(x) \in \mathbb{R}[x]_{\deg\leq 2}$, $\operatorname{coord}_B(p(x)) \in \mathbb{R}^3$ is the coordinates of $p(x)$ with respect to the basis B For instance

$$\operatorname{coord}_U(x + x^2) = (0, 1, 1),$$

and

$$\operatorname{coord}_B(x + x^2) = (0, 0, 1)$$

Also recall the agreement every vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is considered as a 3×1 matrix $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, i e , as a column vector

(a) Find the matrix $\operatorname{Mat}_{U\to U}(T)$ representing the linear transformation T with respect to the basis U

(b) Verify the equation $\operatorname{Mat}_{U\to U}(T)\operatorname{coord}_U(p(x)) = \operatorname{coord}_U(T(p(x)))$, for any $p(x) \in \mathbb{R}[x]_{\deg\leq 2}$

(c) Obtain the change of basis matrix $\mathbf{P}_{B\to U}$ from B to U

(d) Hence, or otherwise, obtain $\operatorname{coord}_B(p(x))$ for each $p(x) \in \mathbb{R}[x]_{\deg\leq 2}$

[Marks for Question 4 $4 + 4 + 2 + 10 = 20$ marks]

Question 5. (a) Given a vector space V, when is a subset $T \subseteq V$ of V said to be a vector subspace of V?

(b) Given a linear transformation $V \xrightarrow{T} W$ show that the kernel $\operatorname{Ker} f$ of f is a vector subspace of V

(c) Given any linear transformation $V \xrightarrow{T} W$ show that the subset $\operatorname{Im}[f] = \{y \in W \quad y = f(x), \text{ for some } x \in V\}$ is a vector subspace of W

(d) Given any linear transformation $V \xrightarrow{f} W$, show that f is one-to-one, if and only if, its kernel is trivial, i e , $\operatorname{Ker} f = \{0\}$

(e) Given a linear transformation $V \xrightarrow{f} W$ between finite dimensional vector spaces, show that

$$\dim \operatorname{Ker} f + \dim \operatorname{Im}[f] = \dim V$$

Hint: Show $\operatorname{Ker} f$ is finite dimensional and then starting with a basis B_0 for $\operatorname{Ker} f$ extend to a basis of V

[Marks for Question 5 $1 + 6 \times 4 = 25$ marks]

Question 6. (a) Given any real symmetric square matrix A of order n, show that the function $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{I} \mathbb{R}^n$ defined by $I(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$ is a symmetric bilinear function

Give examples of matrices A for which the corresponding I is **not** an inner product

State the extra properties on A which would make the corresponding I an inner product

(b) Show that $\mathbb{R}[x]_{\text{deg}\leq 2} \times \mathbb{R}[x]_{\text{deg}\leq 2} \xrightarrow{I} \mathbb{R}[x]_{\text{deg}\leq 2}$ defined by $I(p(x), q(x)) = p(0)q(0)+p(\frac{1}{2})q(\frac{1}{2})+p(1)q(1)$ is an inner product on $\mathbb{R}[x]_{\text{deg}\leq 2}$

Use Gram-Schmidt procedure to obtain a orthonormal basis of $\mathbb{R}[x]_{\text{deg}\leq 2}$ (with respect to the inner product defined above) which contains the polynomial x

[Marks for Question 6 $(6 + 2 + 2) + (5 + 5) = 20$ marks]

TOTAL: [110]

First examiner Dr Partha Pratim Ghosh
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