Tutorial letter 101/3/2018

LINEAR ALGEBRA

MAT2611

Semesters 1 & 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains important information about your module.

BARCODE

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UNISA university of south africa

CONTENTS

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1 INTRODUCTION

Dear Student

Welcome to the MAT2611 module in the Department of Mathematical Sciences at Unisa. We trust that you will find this module both interesting and rewarding.

Some of this tutorial matter may not be available when you register. Tutorial matter that is not available when you register will be posted to you as soon as possible, but is also available on *my*Unisa.

***my*Unisa**

You must be registered on *my*Unisa (`http://my.unisa.ac.za`) to be able to submit assignments online, gain access to the library functions and various learning resources, download study material, "chat" to your lecturers and fellow students about your studies and the challenges you encounter, and participate in online discussion forums. *my*Unisa provides additional opportunities to take part in activities and discussions of relevance to your module topics, assignments, marks and examinations.

Tutorial matter

A tutorial letter is our way of communicating with you about teaching, learning and assessment. You will receive a number of tutorial letters during the course of the module. This particular tutorial letter contains important information about the scheme of work, resources and assignments for this module as well as the admission requirements for the examination. We urge you to read this and subsequent tutorial letters carefully and to keep it at hand when working through the study material, preparing and submitting the assignments, preparing for the examination and addressing queries that you may have about the course (course content, textbook, worked examples and exercises, theorems and their applications in your assignments, tutorial and textbook problems, etc.) to your MAT2611 lecturers.

2 PURPOSE AND OUTCOMES FOR THE MODULE

2.1 Purpose

This module is a direct continuation of MAT1503. It will be useful to students interested in developing their Linear Algebra techniques and skills in solving problems in the mathematical sciences.

2.2 Outcomes

To understand, compute and apply the following linear algebra concepts:

2.2.1 Vector spaces
(Anton & Rorres, sections 4.1 – 4.5), (Lay, sections 4.1 – 4.5).

2.2.2 Rank of a matrix
(Anton & Rorres, sections 4.7 – 4.8), (Lay, section 4.6).

2.2.3 Change of basis
(Anton & Rorres, section 4.6), (Lay, section 4.7).

2.2.4 Eigenvalues and eigenvectors
(Anton & Rorres, section 5.1), (Lay, sections 5.1 – 5.2).

2.2.5 Diagonalisation of matrices
(Anton & Rorres, section 5.2), (Lay, section 5.3).

2.2.6 Inner products and orthogonality
(Anton & Rorres, sections 6.1 – 6.2), (Lay, sections 6.1 – 6.3).

2.2.7 Gram-Schmidt algorithm
(Anton & Rorres, section 6.3), (Lay, section 6.4).

2.2.8 Orthogonal diagonalisation of symmetric matrices
(Anton & Rorres, sections 7.1 – 7.2), (Lay, section 7.1).

2.2.9 Linear transformations
(Anton & Rorres, chapter 8), (Lay, section 4.2 and section 5.4).

3 LECTURER(S) AND CONTACT DETAILS

3.1 Lecturer(s)

The contact details for the lecturer responsible for this module is

Postal address: The MAT2611 Lecturers
Department of Mathematical Sciences
Private Bag X6
Florida
1709
South Africa

Additional contact details for the module lecturers will be provided in a subsequent tutorial letter.

All queries that are not of a purely administrative nature but are about the content of this module should be directed to your lecturer(s). Tutorial letter 301 will provide additional contact details for your lecturer. Please have your study material with you when you contact your lecturer by telephone. If you are unable to reach us, leave a message with the departmental secretary. Provide your name, the time of the telephone call and contact details. If you have problems with questions that you are unable to solve, please send your own attempts so that the lecturers can determine where the fault lies.

Please note: Letters to lecturers may not be enclosed with or inserted into assignments.

3.2 Department

The contact details for the Department of Mathematical Sciences are:

Departmental Secretary:	(011) 670 9147 (SA) +27 11 670 9147 (International)

3.3 University

If you need to contact the University about matters not related to the content of this module, please consult the publication *Study @ Unisa* that you received with your study material. This booklet contains information on how to contact the University (e.g. to whom you can write for different queries, important telephone and fax numbers, addresses and details of the times certain facilities are open). Always have your student number at hand when you contact the University.

4 RESOURCES

4.1 Prescribed books

Prescribed books can be obtained from the University's official booksellers. If you have difficulty locating your book(s) at these booksellers, please contact the Prescribed Books Section at (012) 429 4152 or e-mail vospresc@unisa.ac.za.

Your prescribed textbook for this module is:

Title:	Elementary Linear Algebra with Supplemental Applications
Authors:	Howard Anton and Chris Rorres
Edition:	11th Edition, International Student Version
Publishers:	Wiley
ISBN:	978-1-118-67745-2

However, you may wish to use your copy of

Title:	Linear Algebra and Its Applications
Author:	David C. Lay
Edition:	Pearson New International Edition, 4th edition
Publishers:	Pearson
ISBN:	9781292020556

Students with the textbook by Lay will be accommodated.

Please buy the textbook as soon as possible since you have to study from it directly – you cannot do this module without the prescribed textbook.

4.2 Recommended books

The book "Linear Algebra" by Jim Hefferon is available for free from

`http://joshua.smcvt.edu/linearalgebra/`

with answers to exercises available from the same web site. The concepts are arranged differently to the prescribed book. The relevant chapters and sections are: chapter 2, chapter 3 I-III and V. Some of the terminology is different to the prescribed book.

The book “A First Course in Linear Algebra” by Robert A. Beezer is a free and interactive online book available at

http://linear.ups.edu/

and also has multiple PDF versions available for download. The relevant chapters are “Vectors”, “Matrices” - “Column and Row Spaces”, “Vector Spaces”, “ Eigenvalues”, “Linear Transformations” and “Representations”. Please note that this book assumes that vector spaces are over the field of complex numbers, while the prescribed text book considers only the real numbers.

Finally, the “Book of Proof” (Second Edition) by Richard Hammack, Part I, Chapter 1 (Sets) is recommended for students who need to revise basic set theory and notation. The entire book is available for free from

http://www.people.vcu.edu/~rhammack/BookOfProof/index.html

4.3 Electronic reserves (e-Reserves)

There are no e-Reserves for this module.

4.4 Library services and resources information

For brief information go to:
http://www.unisa.ac.za/brochures/studies

For more detailed information, go to the Unisa website: http://www.unisa.ac.za/, click on **Library**. For research support and services of Personal Librarians, go to:
http://www.unisa.ac.za/sites/corporate/default/Library/Library-services/Research-support

The Library has compiled numerous library guides:

- find recommended reading in the print collection and e-reserves
  - http://libguides.unisa.ac.za/request/undergrad
- request material
  - http://libguides.unisa.ac.za/request/request
- postgraduate information services
  - http://libguides.unisa.ac.za/request/postgrad
- finding , obtaining and using library resources and tools to assist in doing research
  - http://libguides.unisa.ac.za/Research_Skills
- how to contact the Library/find us on social media/frequently asked questions
  - http://libguides.unisa.ac.za/ask

5 STUDENT SUPPORT SERVICES

For information on the various student support services available at Unisa (e.g. student counseling, tutorial classes, language support), please consult the publication *Study @ Unisa* that you received with your study material.

6 STUDY PLAN

The following table provides an outline of the outcomes and ideal dates of completion, and other study activities.

	Semester 1	Semester 2
Outcomes 2.2.1–2.2.3 to be achieved by	17 February 2018	3 August 2018
Outcomes 2.2.3–2.2.5 to be achieved by	16 March 2018	31 August 2018
Outcomes 2.2.6–2.2.9 to be achieved by	13 April 2018	28 September 2018

See the brochure *Study @ Unisa* for general time management and planning skills.

7 PRACTICAL WORK AND WORK INTEGRATED LEARNING

There are no practicals for this module.

8 ASSESSMENT

8.1 Assessment criteria

Specific outcome 1: Understand and apply the definition of a general real vector space, along with the concepts of subspace, linear independence, basis and dimension, row space column space and null space, rank and nullity.

Assessment criteria

You must be able to do the following.

- Decide, with reasons, whether a given set with two given operations defines a vector space.
- Decide, with reasons, whether a given subset of a vector space defines a subspace.
- Find the span of a given set of vectors. Show that a given set of vectors do/do not span a given space, with reasons.
- Test a given set of vectors for linear dependence/independence.
- Find a basis for a given vector space. Find a basis for the span of a given set of vectors. Determine whether or not a given set of vectors forms a basis for a given vector space.
- Find for a given matrix the row space/column space and null space.

- Find, for a given linear system, the general solution.
- Find the rank and nullity of a given matrix.

Specific outcome 2: Understand and be able to apply the basis concepts of inner product spaces.

Assessment criteria

You must be able to do the following.

- Calculate inner products in cases other than the dot product.
- Use the length, angle and distance formulas for arbitrary inner products.
- Test vectors for orthogonality.
- Find orthogonal complements of subspaces.
- Test sets of vectors for orthogonality/orthonormality.
- Use the Gram-Schmidt process to change a basis to an orthogonal/orthonormal one.
- Find the transition matrix between two different bases.
- Find the coordinate vector of a vector with respect to a new basis.
- Decide whether or not a matrix is orthogonal.

Specific outcome 3: Understand and be able to apply the basis concepts of eigenvalues and eigenvectors.

Assessment criteria

You must be able to do the following.

- Test whether a given scalar/vector pair is an eigenvalue–eigenvector pair of a matrix.
- Find the eigenvalues and eigenvectors of a matrix.
- Determine whether or not a given matrix is diagonalisable, giving reasons.
- Find, for a diagonalisable matrix, a diagonalising matrix.
- Determine whether or not a given matrix is orthogonally diagonalisable, giving reasons.
- Find, for an orthogonally diagonalisable matrix, an orthogonal diagonalising matrix.

Specific outcome 4: Understand and be able to apply the concept of linear transformation.

Assessment criteria

The student must be able to:

- Decide, with reasons, whether a given operation on vector space is a linear transformation or not.
- Find the kernel and range of a linear transformation.
- Find the rank and nullity of a linear transformation.
- Determine whether a given linear transformation is one-to-one and/or onto.
- Find, in those cases where is is possible, the inverse of a linear transformation.
- Find the matrix of a linear transformation with respect to a given basis.
- Find the matrices of compositions of transformations and inverse transformations with respect to a given basis.
- Find the matrix of a linear transformation with respect to a basis, given the matrix with respect to a different basis.
- Find the eigenvalues of a linear operator.
- Decide if a given linear transformation is an isomorphism or not, with reasons.

8.2 Assessment plan

A final mark of at least 50% is required to pass the module. If a student does not pass the module then a final mark of at least 40% is required to permit the student access to the supplementary examination. The final mark is composed as follows:

Year mark	
Assignment 01:	30%
Assignment 02:	40%
Assignment 03:	30%

$\longrightarrow$

Final mark	
Year mark:	20%
Exam mark:	80%

8.3 Assignment numbers

8.3.1 General assignment numbers

The assignments for this module are Assignment 01, Assignment 02, etc.

8.3.2 Unique assignment numbers

Please note that each assignment has a unique assignment number which must be written on the cover of your assignment.

8.3.3 Assignment due dates

The dates for the submission of the assignments are:

Semester 1

Assignment 01:	Friday, 23 February 2018
Assignment 02:	Friday, 23 March 2018
Assignment 03:	Friday, 20 April 2018

Semester 2

Assignment 01:	Friday, 10 August 2018
Assignment 02:	Friday, 7 September 2018
Assignment 03:	Friday, 5 October 2018

8.4 Submission of assignments

You may submit written assignments either by post or electronically via *my*Unisa. Assignments may **not** be submitted by fax or e-mail.

For detailed information on assignments, please refer to the *Study @ Unisa* brochure which you received with your study package.

Please make a copy of your assignment before you submit!

To submit an assignment via *my*Unisa:

- Go to *my*Unisa.
- Log in with your student number and password.
- Select the module.
- Click on "Assignments" in the menu on the left-hand side of the screen.
- Click on the assignment number you wish to submit.
- Follow the instructions.

8.5 The assignments

Please make sure that you submit the correct assignments for the 1st semester, 2nd semester or year module for which you have registered. For each assignment there is a **fixed closing date**, the date at which the assignment must reach the University. Late assignments **will not** be marked! When appropriate, solutions for each assignment will be made available on *my*Unisa.
Note that at least one assignment must reach us before the due date in order to gain admission to the examination.

8.6 Other assessment methods

There are no other assessment methods for this module.

8.7 The examination

During the relevant semester, the Examination Section will provide you with information regarding the examination in general, examination venues, examination dates and examination times. For general information and requirements as far as examinations are concerned, see the brochure *Study @ Unisa*.

Registered for ...	Examination period	Supplementary examination period
1st semester module	May/June 2018	October/November 2018
2nd semester module	October/November 2018	May/June 2019
Year module	October/November 2018	January/February 2019

9 FREQUENTLY ASKED QUESTIONS

The *Study @ Unisa* brochure contains an A–Z guide of the most relevant study information.

10 IN CLOSING

We hope that you will enjoy MAT2611 and we wish you all the best in your studies at Unisa!

ADDENDUM A: ASSIGNMENTS – FIRST SEMESTER

ASSIGNMENT 01
Due date: Friday, 23 February 2018
UNIQUE ASSIGNMENT NUMBER: 662670

ONLY FOR SEMESTER 1

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Consider the set

$$X := \left\{ \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \; : \; a \in \mathbb{R} \right\} \subset M_{22}$$

and the operations $\Big($for all $k, a, b \in \mathbb{R}$, $\mathbf{u} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X$ and $\mathbf{v} = \begin{bmatrix} b & 1 \\ 0 & -b \end{bmatrix} \in X\Big)$

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{u} \equiv k \cdot \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} := \begin{bmatrix} ka & 1 \\ 0 & -ka \end{bmatrix},$$

$$+ : X \times X \to X, \qquad \mathbf{u} + \mathbf{v} \equiv \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & 1 \\ 0 & -b \end{bmatrix} := \begin{bmatrix} a+b & 1 \\ 0 & -(a+b) \end{bmatrix}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. The zero vector for X is

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
5. None of the above.

Question 2

Which of the following are subspaces of P_2 with the usual operations ?

A. $\operatorname{span}\{1, x^2\}$

B. $\{a + ax \; : \; a \in \mathbb{R}\}$

C. $\left\{ a + \frac{1}{b}x \, : \, a, b \in \mathbb{R}, \, b \neq 0 \right\}$

D. $\{ ax^3 \, : \, a \in \mathbb{R} \}$

Select from the following:

1. All of A, B, C and D.
2. Only A, B, and D.
3. Only A and B.
4. Only B and D.
5. None of the above.

Question 3

Which of the following sets are linearly independent?

A. $\{ (1, 0), \, (1, 1), \, (1, -1) \}$ in $\mathbb{R}^2$

B. $\{ (1, 1, 1), \, (1, -1, 1), \, (-1, 1, 1) \}$ in $\mathbb{R}^3$

C. $\{ 1 + x, \, x, \, 2 + 3x \}$ in P_2

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A and C.
2. Only B.
3. Only D.
4. Only B and D.
5. None of the above.

Question 4

Which of the following sets are a basis for the following vector subspace of $\mathbb{R}^3$?

$$X = \left\{ \mathbf{x} \in \mathbb{R}^3 \, : \, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{x} \right\}.$$

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ C. $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ D. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A and B.
2. Only B and C.
3. Only C and D.
4. Only A and D.
5. None of the above.

Question 5

Which of the following statements are true:

A. $\dim(\operatorname{span}\{\,(1,0,1),(1,0,-1)\,\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\operatorname{span}\{\,(1,0,0),(-1,0,0)\,\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\operatorname{span}\{\,(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\,\}) = 4$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B and C.
2. Only A and B.
3. Only A and C.
4. Only A.
5. None of the above.

Question 6

Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$?

A. $\{\,\begin{bmatrix} 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \end{bmatrix}\,\}$

B. $\{\,\begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\,\}$

C. $\{\,\begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}\,\}$

D. $\{\,\begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}\,\}$

Select from the following:

1. Only A.
2. Only B, C and D.
3. Only B and C.
4. All of A, B, C and D.
5. None of the above.

Question 7

Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 \end{bmatrix}^T \right\}$

Select from the following:

1. Only B.
2. Only D.
3. Only B and C.
4. Only A.
5. None of the above.

Question 8

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrices A ?

A. $\text{rank}(A) = \text{rank}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A^T) = m$

C. $\text{rank}(A^T) + \text{nullity}(A^T) = n$

D. $\text{row space}(A) = \text{column space}(A)$

Select from the following:

1. Only A and C.
2. Only A and B.
3. Only B.
4. Only C.
5. None of the above.

– End of assignment –

ASSIGNMENT 02
Due date: Friday, 23 March 2018
Total Marks: 40
UNIQUE ASSIGNMENT NUMBER: 733439

ONLY FOR SEMESTER 1

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Question 1: 20 Marks

Let

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

be two bases for span(B_1), where the usual left to right ordering is assumed.

(1.1) Find the transition matrix (change of coordinate/change of basis matrix) $P_{B_1 \to B_2}$. (8)

(1.2) Let B_3 be a basis for $\mathbb{R}^3$ and let the transition matrix from B_2 to B_3 be given by

$$P_{B_2 \to B_3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Find the transition matrix $P_{B_1 \to B_3}$. (6)

(b) Use $P_{B_2 \to B_3}$ to find B_3. (6)

Question 2: 20 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2.1) Determine the characteristic equation for A in λ. (4)

(2.2) Find the eigenvalues of A, and their algebraic multiplicities. (4)

(2.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue. (12)

– End of assignment –

ASSIGNMENT 03
Due date: Friday, 20 April 2018
UNIQUE ASSIGNMENT NUMBER: 668788

ONLY FOR SEMESTER 1

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Let A be an $n \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. $\lambda = 0$.
2. λ is not an eigenvalue of A.
3. $\lambda = 0$ and 0 is an eigenvalue of A.
4. A is invertible.
5. None of the above.

Question 2

Let A be an $n \times n$ matrix with eigenvalue -1, I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Which of the following are true?

A. $(-1)^k$ is an eigenvalue of A^k for all $k \in \mathbb{N}$.

B. $I_n + A$ is singular.

C. $I_n + A = 0_n$.

D. If $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = -\mathbf{x}$, then $\mathbf{x} = 0$.

Select from the following:

1. Only A, B and C.
2. Only A and B.
3. Only A and D.
4. Only B, C and D.
5. None of the above.

Question 3

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B, C and D.
5. None of the above.

Question 4

Let A and B be $n \times n$ matrices and let I_n be the $n \times n$ identity matrix. Then which of the following is always true:

1. $AB + B^T A^T$ is diagonalizable.
2. If A is invertible then A is diagonalizable.
3. If A and B are diagonalizable then $A + B$ is diagonalizable.
4. If $\lambda = 0$ is and eigenvalue of A, then A is not diagonalizable.
5. None of the above.

Select from the following:

1. Only 1.
2. Only 2.
3. Only 3 and 4.
4. Only 2, 3 and 4.
5. None of the above.

Question 5

Which one of the following defines an inner product?

1. $\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} AB^T\right)$ in M_{22}.
2. $\langle a_1 + b_1x + c_1x^2, a_2 + b_2x + c_2x^2\rangle = a_1b_1 + a_2b_2$ in P_2.
3. $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + 2x_2y_2$ in $\mathbb{R}_2$.
4. $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + 2x_2y_2 - 1$ in $\mathbb{R}_2$.
5. None of the above.

Question 6

Which of the following vectors are unit vectors with respect to the inner product $\langle (x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = 2x_1y_1 + 2x_2y_2 + 2x_3y_3$ in $\mathbb{R}^3$?

A. $(1, 0, 0)$ B. $(1, 0, 0)/\sqrt{2}$ C. $(1, 0, 1)/\sqrt{2}$ D. $(1, 1, 0)/2$

Select from the following:

1. Only A.
2. Only B and D.
3. Only A and C.
4. All of A, B, C and D.
5. None of the above.

Question 7

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} A^TB\right)$ in M_{22} ?

A. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. B. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. All of A, B, C and D are orthogonal to each other.
2. None of A, B, C and D are orthogonal to each other.
3. Only A and B are orthogonal, A and C are orthogonal, B and C are orthogonal.
4. Only A and C are orthogonal, B and C are orthogonal.
5. None of the above.

Question 8

Consider the vector subspace $W = \text{span}\{\,1 - x,\ 2x^2\,\}$ of P_2 with the *evaluation inner product* at 0, 1 and -1 (sample points). Which of the following vectors in P_2 lie in the subspace $W^\perp$ of P_2?

1. $x^2 - 1$.
2. $x^2 + x + 1$.
3. x.
4. $-2x^2 + x + 2$.
5. None of the above.

– End of assignment –

ADDENDUM B: ASSIGNMENTS – SECOND SEMESTER

ASSIGNMENT 01
Due date: Friday, 10 August 2018
UNIQUE ASSIGNMENT NUMBER: 712572

ONLY FOR SEMESTER 2

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Consider the set

$$X := \left\{ \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \,:\, a \in \mathbb{R} \right\} \subset M_{22}$$

and the operations $\left(\text{for all } k, a, b \in \mathbb{R},\ \mathbf{u} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X \text{ and } \mathbf{v} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X\right)$

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{u} \equiv k \cdot \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} := \begin{bmatrix} ka & 1 \\ 0 & -ka \end{bmatrix},$$

$$+ : X \times X \to X, \qquad \mathbf{u} + \mathbf{v} \equiv \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & 1 \\ 0 & -b \end{bmatrix} := \begin{bmatrix} a+b & 1 \\ 0 & -(a+b) \end{bmatrix}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements are true in this vector space?

1. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$
2. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$
3. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
4. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$
5. None of the above.

Question 2

Which of the following are subspaces of P_2 with the usual operations ?

A. span $\{1,\ x^2\}$

B. $\{\, 1 + ax \;:\; a \in \mathbb{R} \,\}$

C. $\{\, a - bx^2 \;:\; a, b \in \mathbb{R} \,\}$

D. $\{\, a \;:\; a \in \mathbb{R},\ a \geq 0 \,\}$

Select from the following:

1. Only A, B and C.
2. Only A, C and D.
3. Only C and D.
4. Only A and C.
5. None of the above.

Question 3

Which of the following sets are linearly independent?

A. $\{\, (1, 0),\ (1, 1),\ (1, -1) \,\}$ in $\mathbb{R}^2$

B. $\{\, (1, 1, 1),\ (1, -1, 1),\ (2, -3, 2) \,\}$ in $\mathbb{R}^3$

C. $\{\, 1 + x,\ x,\ 2 + 3x \,\}$ in P_2

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A, B and C.
2. Only B and C.
3. Only B and D.
4. Only D.
5. None of the above.

Question 4

Which of the following sets are a basis for the following vector subspace of M_{22}:

$$X = \left\{ A \in M_{22} \;:\; A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

A. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A and B.
2. Only B and C.
3. Only C and D.
4. Only A and D.
5. None of the above.

Question 5

Which of the following statements are true:

A. $\dim(\operatorname{span}\{1+x^2, 1-x^2\}) = 2$ in P_2

B. $\dim(\operatorname{span}\{x^2, -x^2\}) = 2$ in P_2

C. $\dim(\operatorname{span}\{1+x+x^2, 1+x-x^2, 1-x+x^2, -1+x+x^2\}) = 4$ in P_2

Select from the following:

1. All of A, B, and C.
2. Only A and C.
3. Only A and B.
4. Only A.
5. None of the above.

Question 6

Which of the following sets are a basis for the column space of $\begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Select from the following:

1. All of A, B, C and D.
2. Only B, C and D.
3. Only A.
4. Only B and C.
5. None of the above.

Question 7

Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T, \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 \end{bmatrix}^T \right\}$

Select from the following:

1. Only A.
2. Only C.
3. Only B.
4. Only A.
5. None of the above.

Question 8

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrices A ?

A. $\text{rank}(A) = \text{rank}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A^T) = m$

C. $\text{rank}(A^T) + \text{nullity}(A^T) = n$

D. $\text{row space}(A) = \text{column space}(A)$

Select from the following:

1. Only A and B.
2. Only A and C.
3. Only C.
4. Only A.
5. None of the above.

– End of assignment –

ASSIGNMENT 02
Due date: Friday, 7 September 2018
Total Marks: 40
UNIQUE ASSIGNMENT NUMBER: 822759

ONLY FOR SEMESTER 2

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Question 1: 20 Marks

Let

$$B_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \quad \text{and} \quad B_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

be two bases for span(B_1) in M_{22}, where the usual left to right ordering is assumed.

(1.1) Find the transition matrix (change of coordinate/change of basis matrix) $P_{B_1 \to B_2}$. (8)

(1.2) Let B_3 be a basis for span(B_1) and let the transition matrix from B_2 to B_3 be given by

$$P_{B_2 \to B_3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Find the transition matrix $P_{B_1 \to B_3}$. (6)

(b) Use $P_{B_2 \to B_3}$ to find B_3. (6)

Question 2: 20 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2.1) Determine the characteristic equation for A in λ. (4)

(2.2) Find the eigenvalues of A, and their algebraic multiplicities. (4)

(2.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue. (12)

– End of assignment –

ASSIGNMENT 03
Due date: Friday, 5 October 2018
UNIQUE ASSIGNMENT NUMBER: 762846

ONLY FOR SEMESTER 2

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Let A be an $n \times n$ matrix, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. λ is not an eigenvalue of A.
2. $\lambda = 0$.
3. $\lambda = 0$ and 0 is an eigenvalue of A.
4. A is invertible.
5. None of the above.

Question 2

Let A be an $n \times n$ matrix with eigenvalue -1, I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Which of the following are true?

A. 0 is an eigenvalue of $A + I_n$.

B. $A + I_n$ is singular.

C. $A + I_n = 0_n$.

D. 1 is an eigenvalue of A^2.

Select from the following:

1. Only A, B and D.
2. Only A, B and C.
3. Only A, C and D.
4. All of A, B, C and D.
5. None of the above.

Question 3

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A, C and D.
2. Only A.
3. Only A and B.
4. Only C and D.
5. None of the above.

Question 4

Let A and B be $n \times n$ matrices and let I_n be the $n \times n$ identity matrix. Then

1. If ABB^TA^T is diagonalizable.
2. If A is diagonalizable then A is invertible.
3. If $\lambda = 0$ is an eigenvalue of A, then A is not diagonalizable.
4. If A and B are not diagonalizable then $A + B$ is not diagonalizable.
5. None of the above.

Question 5

Which one of the following defines an inner product?

1. $\langle p(x), q(x) \rangle = p(1)q(1) + 2p(2)q(2) + 3p(3)q(3)$ in P_2.
2. $\langle A, B \rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} AB^T \right)$ in M_{22}.
3. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1$ in $\mathbb{R}_2$.
4. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2 - 1$ in $\mathbb{R}_2$.
5. None of the above.

Question 6

Which of the following vectors are unit vectors with respect to the inner product $\langle (x_1, x_2, x_3), (y_1, y_2.y_3) \rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3$ in $\mathbb{R}^3$?

A. $(1, 0, 0)$ B. $(0, 1, 0)/\sqrt{2}$ C. $(1, 1, 1)/\sqrt{3}$ D. $(1, 1, 0)/2$

Select from the following:

1. Only B and D.
2. Only A, C and D.
3. Only A and C.
4. Only A.
5. None of the above.

Question 7

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B \rangle = \mathrm{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} A^T B \right)$ in M_{22} ?

A. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Select from the following:

1. Only A and C are orthogonal, A and D are orthogonal, C and D are orthogonal.
2. Only B and C are orthogonal.
3. Only A and C are orthogonal, B and D are orthogonal.
4. Only A and C are orthogonal, A and D are orthogonal.
5. None of the above.

Question 8

Consider the vector subspace $W = \mathrm{span}\{\, 1 - x,\, 2x^2 \,\}$ of P_2 with the *standard inner product.* Which of the following vectors in P_2 lie in the subspace $W^{\perp}$ of P_2?

1. $x^2 + 1$.
2. $x + 1$.
3. $x - 1$.
4. $x^2 - 1$.
5. None of the above.

– End of assignment –

ADDENDUM C: EXAM INFORMATION SHEET

The question papers include an information sheet. Please see *my*Unisa for past papers and their information sheets. An example of an information sheet is reproduced below. The information sheet includes all of the essential concepts and theorems.

INFORMATION SHEET

Vector spaces

Definition (Vector space)**.**
A vector space *is a non-empty set* V *with vector addition* $+ : V \times V \to V$ *and scalar multiplication* $\cdot : \mathbb{R} \times V \to V$ *obeying the axioms*

VS1. $\mathbf{u} + \mathbf{v} \in V$ *for all* $\mathbf{u}, \mathbf{v} \in V$*,*

VS2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for all* $\mathbf{u}, \mathbf{v} \in V$*,*

VS3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$*,*

VS4. there exists $\mathbf{0} \in V$ *such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$ *for all* $\mathbf{u} \in V$*,*

VS5. for all $\mathbf{u} \in V$ *there exists* $-\mathbf{u} \in V$ *such that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$*,*

VS6. $a \cdot \mathbf{u} \in V$ *for all* $a \in \mathbb{R}$*,* $\mathbf{u} \in V$*,*

VS7. $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$ *for all* $a \in \mathbb{R}$*,* $\mathbf{u}, \mathbf{v} \in V$*,*

VS8. $(a + b) \cdot \mathbf{u} = a \cdot \mathbf{u} + b \cdot \mathbf{u}$ *for all* $a, b \in \mathbb{R}$*,* $\mathbf{u} \in V$*,*

VS9. $a \cdot (b \cdot \mathbf{u}) = (ab) \cdot \mathbf{u}$ *for all* $a, b \in \mathbb{R}$*,* $\mathbf{u} \in V$*,*

VS10. $1 \cdot \mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in V$*.*

Theorem (VZ)**.** $\mathbf{0} = 0 \cdot \mathbf{u} = a \cdot \mathbf{0}$ *for all* $a \in \mathbb{R}$ *and* $\mathbf{u} \in V$ *in a vector space* V*.*

Theorem (VN)**.** $(-1) \cdot \mathbf{u} = -\mathbf{u}$ *for all* $\mathbf{u} \in V$ *in a vector space* V*.*

Definition (Subspace)**.**
A subset $W \subseteq V$ *of a vector space* V *is a* subspace *of* V *if* W*, with the same vector addition and scalar multiplication as* V*, is a vector space.*

Theorem (SS)**.**
A subset $W \subseteq V$ *of a vector space* V *is a* subspace *of* V*, with the same vector addition* $+$ *and scalar multiplication* $\cdot$ *as* V*, if and only if*

1. W *is not empty,*
2. $\mathbf{u} + \mathbf{v} \in W$ *for all* $\mathbf{u}, \mathbf{v} \in W$*,*

3. $a \cdot \mathbf{u} \in W$ *for all* $a \in \mathbb{R}$, $\mathbf{u} \in V$.

Definition (Linear independence)**.**
A subset $\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subseteq V$ *in a vector space* V *is* linearly independent *if and only if*

$$c_1 \cdot \mathbf{b}_1 + \cdots + c_n \cdot \mathbf{b}_n = \mathbf{0} \quad \iff \quad c_1 = \cdots = c_n = 0.$$

Definition (Span)**.**
The span *of a subset* $\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subseteq V$ *in a vector space* V *is the subspace of* V *given by*

$$\operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_n\} = \{c_1 \cdot \mathbf{b}_1 + \cdots + c_n \cdot \mathbf{b}_n \, : \, c_1, \dots, c_n \in \mathbb{R}\}.$$

Definition (Basis, dimension)**.**
A subset $\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subseteq V$ *in a vector space* V *is a* basis *for* V *if and only if*

1. $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ *is linearly independent,*
2. $\operatorname{span}\{\mathbf{b}_1, \dots, \mathbf{b}_n\} = V$.

If $\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subseteq V$ *is a basis for* V *then the dimension of* V *is* n, $\dim(V) = n$.

Definition (Coordinate matrix)**.**
Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ *be a basis for* V *and let* $\mathbf{v} \in V$*. Then there exists unique* $c_1, \dots, c_n \in \mathbb{R}$ *such that* $\mathbf{v} = c_1 \cdot \mathbf{b}_1 + \cdots + c_n \cdot \mathbf{b}_n$*. The column vector*

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the coordinate matrix *of* $\mathbf{v}$ *relative to* B.

Definition (Transition matrix, change of coordinate matrix)**.**
Let $B_1 = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ *be a basis for the vector space* V*, and* B_2 *be another basis for* V*. The* transition matrix (change of coordinate matrix) $P_{B_1 \to B_2}$ *from* B_1 *to* B_2 *is given by*

$$P_{B_1 \to B_2} = \begin{bmatrix} [\mathbf{b}_1]_{B_2} & \cdots & [\mathbf{b}_n]_{B_2} \end{bmatrix}.$$

Examples (of vector spaces)**.**

- $\mathbb{R}^n$
- *The vector space* $P_n = \{c_0 + c_1 x + \cdots + c_n x^n \, : \, c_0, \dots, c_n \in \mathbb{R}\}$ *of polynomials of degree* n *or less.*
- *The vector space* M_{mn} *of* $m \times n$ *matrices.*

Inner products

Definition (Inner product)**.**
An inner product *is a function* $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ *on a vector space* V *which obeys the axioms*

IP1. $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$ *for all* $\mathbf{u}, \mathbf{v} \in V$*,*

IP2. $\langle k \cdot \mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle$ *for all* $k \in \mathbb{R}$*,* $\mathbf{u}, \mathbf{v} \in V$*,*

IP3. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$*,*

IP4. a) $\langle \mathbf{u}, \mathbf{u}\rangle \geq 0$*, for all* $\mathbf{u} \in V$*,*
b) $\langle \mathbf{u}, \mathbf{u}\rangle = 0$ *if and only if* $\mathbf{u} = 0$*.*

Definition (Orthogonality)**.**
Let $\langle\cdot,\cdot\rangle$ *denote an inner product on a vector space* V*. If* $\langle \mathbf{u}, \mathbf{v}\rangle = 0$*, then* $\mathbf{u}$ *and* $\mathbf{v}$ *are* orthogonal *to each other.*

Definition (Unit vector, normalized)**.**
Let $\langle\cdot,\cdot\rangle$ *denote an inner product on a vector space* V*. If* $\langle \mathbf{u}, \mathbf{u}\rangle = 1$*, then* $\mathbf{u}$ *is a* unit vector (normalized)*.*

Theorem (Cauchy-Schwarz inequality)**.**
Let $\langle\cdot,\cdot\rangle$ *denote an inner product on a vector space* V*. Then*

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle}$$

for all $\mathbf{u}, \mathbf{v} \in V$*.*

Definition (Gram-Schmidt process)**.**
Let $\langle\cdot,\cdot\rangle$ *denote an inner product on a vector space* V *and let* $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *be a linearly independent set in* V*. The* Gram-Schmidt process *yields an* orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *for* $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *as follows*

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1, \\
&\vdots \\
\mathbf{v}_m &= \mathbf{u}_m - \frac{\langle \mathbf{u}_m, \mathbf{v}_1\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{u}_m, \mathbf{v}_{m-1}\rangle}{\langle \mathbf{v}_{m-1}, \mathbf{v}_{m-1}\rangle}\mathbf{v}_{m-1}.
\end{aligned}$$

An orthonormal basis $\{\mathbf{v}'_1, \ldots, \mathbf{v}'_m\}$ *is obtained by setting* $\mathbf{v}'_j = \dfrac{\mathbf{v_j}}{\langle \mathbf{v}_j, \mathbf{v}_j\rangle}$*.*

Linear transformations

Definition (Linear transformation)**.**
A function $T : V \to W$ *between vector spaces* V *and* W *is a* linear transformation *if and only if*

1. $T(k \cdot \mathbf{u}) = k \cdot T(\mathbf{u})$ *for all* $k \in \mathbb{R}$, $\mathbf{u} \in V$
2. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in V$

Examples (of linear transformations)**.**

- *The trace operation on* M_{nn} *is a linear transformation* $\mathrm{tr} : M_{nn} \to \mathbb{R}$.
- *The transpose operation on* M_{mn} *is a linear transformation.*

Definition (Kernel, nullity)**.**
The kernel *of a linear transformation* $T : V \to W$ *between vector spaces* V *and* W *is the subspace*

$$\ker(T) = \{\mathbf{v} \in V \,:\, T(v) = \mathbf{0}_W\}$$

of V*, where* $\mathbf{0}_W$ *is the zero vector in* W*. The* nullity *of* T *is the dimension of* $\ker(T)$.

Definition (Range, rank)**.**
The range *of a linear transformation* $T : V \to W$ *between vector spaces* V *and* W *is the subspace*

$$R(T) = \{T(\mathbf{v}) \,:\, \mathbf{v} \in V\}$$

of W*. The* rank *of* T *is the dimension of* $R(T)$.

Definition (One-to-one, injective, inverse)**.**
A linear transformation $T : V \to W$ *between vector spaces* V *and* W *is* one-to-one *if and only if*

$$T(\mathbf{u}) = T(\mathbf{v}) \quad \Longleftrightarrow \quad \mathbf{u} = \mathbf{v}.$$

A one-to-one linear transformation $T : V \to W$ *has an inverse linear transformation* $T^{-1} : R(T) \to V$ *satisfying* $T^{-1}(T(\mathbf{u})) = \mathbf{u}$ *for all* $\mathbf{u} \in V$.

Definition (Onto, surjective)**.**
A linear transformation $T : V \to W$ *between vector spaces* V *and* W *is* onto *if and only if* $R(T) = W$.

Theorem (TO)**.** *If* V *and* W *are finite dimensional vector spaces and* $T : V \to W$ *is a linear transformation, then* T *is one-to-one if and only if* $\ker(T) = \{\mathbf{0}\}$*. If* $\dim(V) = \dim(W)$*, then* T *is onto if and only if* T *is one-to-one.*

Definition (Isomorphism, bijection)**.**
A one-to-one and onto linear transformation $T : V \to W$ *between vector spaces* V *and* W *is an* isomorphism (bijection)*. If an isomorphism between* V *and* W *exists, then* V *and* W *are* isomorphic.

Theorem (VI). *Every vector space V with $\dim(V) = n$ is isomorphic to $\mathbb{R}^n$.*

Definition (Matrix representation of a linear transformation).
Let $B_V = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V, and B_W be a basis for the vector space W. The matrix representation $[T]_{B_W,B_V}$ *of a linear transformation $T : V \to W$ is given by*

$$[T]_{B_W,B_V} = \begin{bmatrix} [T(\mathbf{b}_1)]_{B_W} & \cdots & [T(\mathbf{b}_n)]_{B_W} \end{bmatrix}.$$

When $V = W$ and $B_V = B_W$, we write $[T]_{B_V} = [T]_{B_V,B_V}$.

Matrices

Definition (Column space, row space, rank).
Let A be an $m \times n$ matrix with columns $A = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix}$ and rows $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$.
The column space *of A is* $\operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ *and the* row space *of A is* $\operatorname{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$. *The* rank *of A is the dimension of the column and row spaces,* $\operatorname{rank}(A) = \dim(\operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}) = \dim(\operatorname{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\})$.

Definition (Null space, nullity).
The null space *of an $m \times n$ matrix A is the subspace*

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = \mathbf{0}\}.$$

The nullity *of T is the dimension of $N(A)$.*

Theorem (RN). $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$ *for every $m \times n$ matrix A.*

Definition (Eigenvalue, eigenvector).
Let A be an $n \times n$ matrix. If $A\mathbf{x} = \lambda\mathbf{x}$, for $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$, then λ is an eigenvalue *of A and $\mathbf{x}$ is an* eigenvector *of A corresponding to the eigenvalue λ.*

Definition (Eigenspace, geometric multiplicity).
Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. Then

$$E_\lambda = \{\mathbf{x} \in \mathbb{C}^n \,:\, A\mathbf{x} = \lambda\mathbf{x}\}$$

is a vector space, called the eigenspace *for the eigenvalue λ of A. The* geometric multiplicity *of λ is* $\dim(E_\lambda)$.

Definition (Characteristic equation, characteristic polynomial).
Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ satisfies the characteristic equation $\det(\lambda I_n - A) = 0$, *where I_n is the $n \times n$ identity matrix. The polynomial* $\det(xI_n - A)$ *is the* characteristic polynomial *in the variable x.*

Definition (Algebraic multiplicity).
Let A *be an* $n \times n$ *matrix with eigenvalue* λ*. The* algebraic multiplicity *of* λ *is the largest number* $a \in \mathbb{N}$ *such that* $(x-\lambda)^a$ *is a factor of the characteristic polynomial* $\det(xI_n - A)$*.*

Definition (Diagonalizable).
An $n \times n$ *matrix* A *is* diagonalizable *if and only if* A *is similar to some* $n \times n$ *diagonal matrix* D*, i.e.* $A = PDP^{-1}$ *for some* $n \times n$ *diagonal matrix* D *and non-singular* $n \times n$ *matrix* P*.*

Theorem (DI)**.** *An* $n \times n$ *matrix* A *is diagonalizable if and only if* A *has* n *linearly independent eigenvectors.*

Theorem (DD)**.** *If an* $n \times n$ *matrix* A *has* n *distinct eigenvalues, then* A *is diagonalizable.*

Theorem (DS)**.** *If an* $n \times n$ *matrix* A *is symmetric, then* A *is diagonalizable.*

Theorem (DM)**.** *For a square matrix* A*, the algebraic and geometric multiplicity are equal for each eigenvalue of* A *if and only if* A *is diagonalizable.*

Definition (Trace).
The trace *of a square matrix is the sum of its diagonal entries*

$$\operatorname{tr}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Theorem (CT)**.** *For all* $n \times n$ *matrices* A*,* B *and* C *we have* $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$*. Consequently* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$*.*

Definition (Transpose).
The transpose of a matrix is obtained by interchanging corresponding rows and columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Theorem (TT)**.** *For all* $m \times n$ *matrices* A *we have* $(A^T)^T = A$*.*

Theorem (TI)**.** *For all* $n \times n$ *matrices* A *we have* $\operatorname{tr}(A) = \operatorname{tr}(A^T)$*.*

Determinants

For 2×2 and 3×3 matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Cofactor expansion along the j-th row:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,k-1} & a_{j-1,k+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,k-1} & a_{j+1,k+1} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{nn} \end{vmatrix}$$

Cofactor expansion along the k-th column:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,k-1} & a_{j-1,k+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,k-1} & a_{j+1,k+1} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{nn} \end{vmatrix}$$

Theorem (DC). *For all* $k \in \mathbb{R}$ *and* $n \times n$ *matrices* A *we have* $\det(kA) = k^n \det(A)$.

Theorem (DP). *For all* $n \times n$ *matrices* A *and* B *we have* $\det(AB) = \det(A)\det(B)$.

ADDENDUM D: USEFUL COMPUTER SOFTWARE

It is possible to check the correctness of your calculations by hand. If you are interested in software that may help to check your results please consult the following resources. **Note however that the software will not be available at exam time, so it is recommended to be proficient at checking your own results by hand**.

Maxima:
`http://maxima.sourceforge.net/`
`http://maxima.sourceforge.net/docs/intromax/intromax.html` (section 6).
`http://maxima.sourceforge.net/docs/manual/en/maxima_23.html`

Maxima is also available for Android devices:
`https://sites.google.com/site/maximaonandroid/`

See addendum E for a brief introduction to Maxima for Linear Algebra.

Wolfram Alpha:
`http://www.wolframalpha.com/`
`http://www.wolframalpha.com/examples/Matrices.html`

Please note that the use of software **is not required** for this module.

ADDENDUM E: ELEMENTARY LINEAR ALGEBRA USING MAXIMA

A complete guide to Maxima is beyond the scope of this module. Here we list only the most essential features. Please consult `http://maxima.sourceforge.net/` for documentation on Maxima.

Please note that the use of software **is not required** for this module.

E.1 The `linearalgebra` and `eigen` packages

First we load the packages `eigen` and `linearalgebra`. Type only the line following `(%i1)` in the white boxes, i.e. `load(eigen);`

```
(%i1) load(eigen);
```

```
(%o1)          /usr/pkg/share/maxima/5.27.0/share/matrix/eigen.mac
```

```
(%i2) load(linearalgebra);
```

```
0 errors, 0 warnings
(%o2) /usr/pkg/share/maxima/5.27.0/share/linearalgebra/linearalgebra.mac
```

The output `(%o1)` and `(%o2)` and may be different, but there should be no error messages. Note the semicolon `;` after every command.

E.2 Matrices

Now we input the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

```
(%i3) A: matrix( [1, 2, 3],
                 [4, 5, 6] );
```

```
                             [ 1  2  3 ]
(%o3)                        [         ]
                             [ 4  5  6 ]
```

```
(%i4) B: matrix( [-1, -2],
                 [ 1,  2],
                 [ 0,  0] );
```

```
                            [ - 1  - 2 ]
                            [          ]
(%o4)                       [  1    2  ]
                            [          ]
                            [  0    0  ]
```

Type carefully to reproduce the input (%i3) and (%i4) correctly. Next we calculate the matrix product $C = AB$. The matrix product is denoted by a full stop between A and B.

```
(%i5) C: A . B;
```

```
                                 [ 1  2 ]
(%o5)                            [      ]
                                 [ 1  2 ]
```

E.3 Eigenvalues and eigenvectors

We can determine the eigenvalues of C, namely 0 and 3 each with algebraic multiplicity 1. The expression eigenvalues(C) returns a list of eigenvalues [0, 3] and a list of multiplicities for each eigenvalue [1, 1] where the multiplicities are in the same order as the eigenvalues.

```
(%i6) eigenvalues(C);
```

```
(%o6)                         [[0, 3], [1, 1]]
```

Similarly the eigenvectors eigenvectors(C) can be determined. This returns three lists, the first two are the same as for eigenvalues(C) while the last is a list of eigenvectors.

```
(%i7) eigenvectors(C);
```

```
                                               1
(%o7)                  [[[0, 3], [1, 1]], [[[1, - -]], [[1, 1]]]]
                                               2
```

i.e. we find the eigenvector

$$\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

for the corresponding eigenvalue 0 of C and the eigenvector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for the corresponding eigenvalue 3 of C. The normalized eigenvectors (uniteigenvectors(C)) can be determined similarly.

```
(%i8) uniteigenvectors(C);
```

```
                              2          1             1         1
(%o8) [[[0, 3], [1, 1]], [[[-------, - -------]], [[-------, -------]]]]
                           sqrt(5)    sqrt(5)       sqrt(2)   sqrt(2)
```

i.e. the normalized eigenvector corresponding to the eigenvalue 0 of C is

$$\begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

Although you may find a different eigenvector, that does not mean your answer is incorrect!

E.4 Rank, nullity, columnspace and nullspace

The rank of A (appears above) is calculated with `rank(A)`, the nullity with `nullity(A)`, the columnspace with `columnspace(A)` and the nullspace with `nullspace(A)`. Once again, your own answers may differ but still be correct!

```
(%i9) rank(A);
```

```
(%o9)                                2
```

```
(%i10) columnspace(A);
```

```
                                 [ 1 ]   [ 2 ]
(%o10)                      span([   ], [   ])
                                 [ 4 ]   [ 5 ]
```

```
(%i11) nullspace(A);
```

```
                                    [ - 3 ]
                                    [     ]
(%o11)                         span([  6  ])
                                    [     ]
                                    [ - 3 ]
```

```
(%i12) nullity(A);
```

```
(%o12)                               1
```

E.5 Matrix inverse

The inverse of a matrix (when it exists) is calculated using `invert`. Here we calculate

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}.$$

```
(%i13) invert(matrix( [1,1], [1,2] ));
```

```
                               [  2    - 1 ]
(%o13)                         [           ]
                               [ - 1    1  ]
```

E.6 Gram-Schmidt algorithm

The Gram-Schmidt algorithm is easily applied using `gramschmidt`. The vectors for which we want to find an orthogonal basis are specified as *rows* of a matrix. For example, below we apply the gram-Schmidt algorithm for

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with respect to the Euclidean inner product.

```
(%i14) gramschmidt(matrix([1,1],[0,1]));
```

```
                                           1   1
(%o14)                     [[1, 1], [- -, -]]
                                           2   2
```

i.e. we find the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}.$$

Now consider a non-Euclidean inner product on $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 + 2 x_2 y_2, \qquad \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \qquad x_1, x_2, y_1, y_2 \in \mathbb{R}$$

```
(%i15) f(x,y):= x[1]*y[1] + 2*x[2]*y[2];
```

```
(%o15)                  f(x, y) := x  y  + 2 x  y
                                    1  1      2  2
```

we can tell `gramschmidt` to use `f` (our inner product) when applying the Gram-Schmidt algorithm

```
(%i16) ob: gramschmidt(matrix([1,1],[0,1]), f);
```

```
                                           2   1
(%o16)                     [[1, 1], [- -, -]]
                                           3   3
```

i.e. we find the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right\}.$$

with respect to our non-Euclidean inner product. To find an orthonormal basis we need to normalize each of these vectors with respect to our non-Euclidean inner product by extracting each vector and divide by its norm. Here we use `first`, `second` and so on to obtain each of the vectors.

```
(%i17) v1: first(ob);
```

```
(%o17)                          [1, 1]
```

```
(%i18) v1 / sqrt(f(v1,v1));
```

```
                                 1        1
(%o18)                       [-------, -------]
                              sqrt(3)  sqrt(3)
```

```
(%i19) v2: second(ob);
```

```
                                      2  1
(%o19)                             [- -, -]
                                      3  3
```

```
(%i20) v2 / sqrt(f(v2,v2));
```

```
                           2              1
(%o20)               [- -------------, -------------]
                             4   2           4   2
                       3 sqrt(- + -)  3 sqrt(- + -)
                             9   9           9   9
```

To simplify the rational expressions, use `ratsimp`.

```
(%i21) ratsimp(v2 / sqrt(f(v2,v2)));
```

```
                         sqrt(2)         1
(%o21)                [- -------, ---------------]
                         sqrt(3)  sqrt(2) sqrt(3)
```

ADDENDUM F: Further Problems

In this section the question papers for the years 2016, 2015 and 2014 are provided. These provide extra resources for problems and provide a general hint towards the standard of problems for the examinations.

QUESTION 1

This question is a **multiple choice** question and should be answered in the **answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book. Write only the *number* for your answer.

(1.1) Consider the set (2)

$$X := \{\spadesuit\}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which of the following statements are true in X ?

A. for all $\mathbf{x} \in X$: $\mathbf{x} - \mathbf{x} = \spadesuit$

B. for all $\mathbf{x} \in X$: $0 \cdot \mathbf{x} = \mathbf{x}$

C. $\mathbf{0} = 0$

D. $\mathbf{0} = (0, 0)$

Choose from the following:

1. C and D
2. A and B
3. Only A
4. C or D
5. None of the above.

(1.2) Which of the following are subspaces of P_2 with the usual operations ? (2)

A. $\{1 + ax : a \in \mathbb{R}\}$

B. span $\{1 + x, 1 + x^2\}$

C. span $\{1, x, x^2, x^3\} = P_3$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B.
4. Only B and C.
5. None of the above.

[TURN OVER]

(1.3) Which of the following sets are linearly independent? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right\}$ in M_{22}

B. $\{ (1,1,-1), (1,-1,-1), (-1,-1,-1) \}$ in $\mathbb{R}^3$

C. $\{ 1-x, 1-x^2, 1-x+x^2 \}$ in P_2

Select from the following:

1. All of A, B and C.
2. Only B and C.
3. Only B.
4. Only C.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Select from the following:

1. All of A, B and C.
2. Only A.
3. Only A and B.
4. Only A and C.
5. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim\left(\text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}\right) = 2$ in M_{22}

B. $\dim(\text{span}\{ (1,1,-1), (1,-1,-1), (-1,-1,-1) \}) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\text{span}\{ 1-x, 1-x^2, x-x^2 \}\right) = 2$ in P_2

Select from the following:

1. Only A.
2. Only B.

[TURN OVER]

3. Only B and C.
4. Only C.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -2 \\ -1 & -1 & 0 & 2 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 0 & 2 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & -2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 1 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Both B and C.
5. None of the above.

(1.7) Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -2 \\ -1 & -1 & 0 & 2 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only D.
5. None of the above.

[TURN OVER]

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -2 \\ -1 & -1 & 0 & 2 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.
2. $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.
3. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
4. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.
5. None of the above.

[16]

QUESTION 2

Consider the vector space M_{22}.

(2.1) Show that (12)

$$\langle A, B \rangle = \text{tr}\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} AB^T \right)$$

is an inner product on M_{22}. You may use that

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\rangle = a^2 + b^2 + c^2 + d^2 + (a+c)^2 + (b+d)^2.$$

(2.2) Suppose that $A \in M_{22}$ is orthogonal to $C \in M_{22}$ and $B \in M_{22}$ is orthogonal to C. Prove that X is orthogonal to C, with respect to the inner product **defined in 2.1** above, for all $X \in \text{span}\{A, B\}$. (5)

(2.3) Apply the Gram-Schmidt process to the following subset of M_{22}: (12)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

(2.4) Let V be a vector space with zero vector $\mathbf{0}$ and let $\langle \cdot, \cdot \rangle$ denote an inner product on V. Prove that $\langle \mathbf{0}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. (4)

[33]

QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

[TURN OVER]

(3.1) Determine the nullity of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda(\lambda - 1)^2 = 0.$$

(3.3) Find bases for the eigenspaces of A. (14)

(3.4) For each eigenvalue, determine the algebraic and geometric multiplicity. Is A diagonalizable? (5)

(3.5) Let B be an $n \times n$ matrix. Prove that BB^T is diagonalizable. (2)

(3.6) Prove or disprove: (3)

If B is a 2×2 matrix with $\det(B) < 0$, then B is diagonalizable.

Note that the characteristic equation for B is $\lambda^2 - \mathrm{tr}(B)\lambda + \det(B) = 0$.

[29]

QUESTION 4

Let $T : \mathbb{R}^3 \to P_2$ be defined by

$$T(a, b, c) = (a + b + c) + (a + b)x + ax^2.$$

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B_2,B_1}$ of T relative to the basis (8)

$$B_1 = \{ (1, 1, -1), (1, -1, -1), (-1, -1, -1) \}$$

in $\mathbb{R}^3$ and the basis

$$B_2 = \{ 1, x, x^2 \}$$

in P_2, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = P_2$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

[22]

TOTAL MARKS: [100]

QUESTION 1

This question is a **multiple choice** question and should be answered in the **answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book. Write only the *number* for your answer.

(1.1) Consider the set (2)

$$X := \{\spadesuit\}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which of the following statements are true in X ?

A. for all $\mathbf{x} \in X$: $-\mathbf{x} = \spadesuit$

B. for all $\mathbf{x} \in X$: $-\mathbf{x} = \mathbf{x}$

C. $\mathbf{0} = 0$

D. $\mathbf{0} = (0, 0)$

Choose from the following:

1. A
2. B
3. A and B
4. C or D
5. None of the above.

(1.2) Which of the following are subspaces of M_{22} with the usual operations ? (2)

A. $\operatorname{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} : a \geq 0 \right\}$

C. $\left\{ \begin{bmatrix} a & -1 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B and C.
4. All of A, B and C.
5. None of the above.

[TURN OVER]

(1.3) Which of the following sets are linearly independent? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in M_{22}

B. $\{ (1,0,1),\ (0,1,0),\ (1,1,-1) \}$ in $\mathbb{R}^3$

C. $\left\{ 1-x,\ 1-x^2,\ 1-x+x^2 \right\}$ in P_2

Select from the following:

1. Only A and C.
2. Only B and C.
3. Only B.
4. Only C.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Both A and B.
2. Only A.
3. Only B.
4. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim\left(\text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}\right) = 2$ in M_{22}

B. $\dim\left(\text{span}\{ (1,0,1),\ (0,1,0),\ (1,1,-1) \}\right) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\text{span}\left\{ 1-x,\ 1-x^2,\ 1-x+x^2 \right\}\right) = 2$ in P_2

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only A and B.
5. None of the above.

[TURN OVER]

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Only C.
5. None of the above.

(1.7) Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both B and C.
4. All of A, B and C.
5. None of the above.

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.
2. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.

[TURN OVER]

3. rank$(A) = 2$, nullity$(A) = 2$.
4. rank$(A) = 2$, nullity$(A) = 1$.
5. None of the above.

[16]

QUESTION 2

Consider the vector space M_{22}.

(2.1) Show that (12)

$$\langle A, B\rangle = \text{tr}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix} AB^T\right)$$

is an inner product on M_{22}.

(2.2) Prove that if $A, B \in M_{22}$, where $A, B \neq \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, are orthogonal to each other with respect to the inner product **defined in 2.1** above, then $\{ A, B \}$ is a linearly independent set. (6)

(2.3) Apply the Gram-Schmidt process to the following subset of M_{22}: (12)

$$\left\{ \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

(2.4) Let V be a vector space with zero vector $\mathbf{0}$ and let $\langle \cdot, \cdot\rangle$ denote an inner product on V. Prove that $\langle \mathbf{0}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in V$. (4)

[34]

QUESTION 3

Consider the matrix

$$A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 1 & 0\end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda(\lambda - 1)^2 = 0.$$

(3.3) Find bases for the eigenspaces of A. (14)

(3.4) For each eigenvalue, determine the algebraic and geometric multiplicity. Is A diagonalizable? (5)

(3.5) Prove or disprove: (2)

If B is a 2×2 matrix, then B is diagonalizable if and only if B^2 is diagonalizable.

(3.6) Let B be an $n \times n$ matrix. Prove that $B + B^T$ is diagonalizable. (2)

[28]

[TURN OVER]

QUESTION 4

Let $T : \mathbb{R}^3 \to M_{22}$ be defined by $T(x, y, z) = \begin{bmatrix} x & y \\ z & x \end{bmatrix}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B_2,B_1}$ of T relative to the basis (8)

$$B_1 = \{ (1, 0, 1), (0, 1, 0), (1, 0, -1) \}$$

in $\mathbb{R}^3$ and the basis

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

[22]

TOTAL MARKS: [100]

QUESTION 1

This question is a **multiple choice** question and should be answered in the **green answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book.

(1.1) Consider the set (2)

$$X := \{ \spadesuit \}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $\mathbf{0} = 0$
2. $\mathbf{0} = \spadesuit$
3. $\mathbf{0} = (0, 0)$
4. $\mathbf{0} = (\spadesuit, \spadesuit)$
5. None of the above.

(1.2) Which of the following are subspaces of M_{22} with the usual operations ? (2)

A. $\operatorname{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,:\, a, b, c, d \in \mathbb{R},\ a = -d \right\}$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B and C.
4. All of A, B and C.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. $\{ (1, 0, 1),\ (0, 0, 0) \}$ in $\mathbb{R}^3$

B. $\{ (1, 0, 1),\ (0, 1, 0),\ (1, 1, 1) \}$ in $\mathbb{R}^3$

C. $\{ 1 - x,\ 1 - x^2 \}$ in P_2

Select from the following:

1. Only A.

[TURN OVER]

2. Only B.
3. Only A and B.
4. Only C.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,0,1), (0,1,0), (1,1,1)\}) = 3$ in $\mathbb{R}^3$

B. $\dim(\{(1,0,1), (0,1,0), (1,1,1)\}) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}\right) = 1$ in M_{22}

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Both A and C.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? (2)

A. $\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \}$

B. $\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Both B and C.
5. None of the above.

(1.7) Which of the following sets are a basis for the column space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both B and C.
4. All of A, B and C.
5. None of the above.

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.
2. $\text{rank}(A) = 2$, $\text{nullity}(A) = 0$.
3. $\text{rank}(A) = 1$, $\text{nullity}(A) = 2$.
4. $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.
5. None of the above.

[16]

[TURN OVER]

QUESTION 2

Consider the vector space P_3.

(2.1) Show that (12)

$$\langle p(x), q(x)\rangle := p_0q_0 + p_1q_1 + p_2q_2 + 3p_3q_3,$$

where

$$p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 \quad \text{and} \quad q(x) = q_0 + q_1x + q_2x^2 + q_3x^3,$$

is an inner product on P_3.

(2.2) Prove that if $p(x), q(x) \in P_3$, where $p(x), q(x) \neq 0$, are orthogonal to each other with respect to the inner product **defined in 2.1** above, then $\{\, p(x),\, q(x)\,\}$ is a linearly independent set. (6)

(2.3) Apply the Gram-Schmidt process to the following subset of P_3: (12)

$$\left\{\, 1 + x^3,\ -1 + x^3,\ -1 + x + x^3 \,\right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

[30]

QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda^2(\lambda - 2) = 0.$$

(3.3) Find bases for the eigenspaces of A. (18)

(3.4) Prove or disprove: (2)

If B is a 2×2 non-singular matrix, then B is diagonalizable.

(3.5) Let B be an $n \times n$ non-singular matrix. Prove that: (5)

B is diagonalizable if and only if B^{-1} is diagonalizable.

[30]

[TURN OVER]

QUESTION 4

Let $T : M_{22} \to M_{22}$ be defined by $T(A) = A + A^T$ where A^T is the transpose of A.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_B$ of T relative to the basis (10)

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22} ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

[**24**]

TOTAL MARKS: [100]

QUESTION 1

This question is a **multiple choice** question and should be answered in the **green answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book.

(1.1) Consider the set (2)

$$X := \{ (x, y) \, : \, x, y \in \mathbb{R} \}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx + k - 1, ky),$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha + 1, y + \beta).$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $-(1, 1) = (-3, -1)$
2. $-(1, 1) = (-2, -1)$
3. $-(1, 1) = (-1, -1)$
4. $-(1, 1) = (0, -1)$
5. None of the above.

(1.2) Which of the following are subspaces of P_1 with the usual operations ? (2)

A. $\text{span}\{ 1 + x \}$

B. $\{ ax \, : \, a \in \mathbb{R} \}$

C. $\{ 1 + ax \, : \, a \in \mathbb{R} \}$

D. $\{ (1 + a)x \, : \, a \in \mathbb{R} \}$

Select from the following:

1. All of A, B, C and D.
2. Only A, B and D.
3. Only A, B and C.
4. Only B, C and D.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. $\text{span}\{ (1, 0, 1), \, (1, 0, 2) \}$ in $\mathbb{R}^3$

B. $\{ (1, 0, 1), \, (1, 0, 2) \}$ in $\mathbb{R}^3$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A.
2. Only A and B.

[TURN OVER]

3. Only B.
4. Only B and C.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of P_2: (2)

$$X = \{\, p(x) \in P_2 \,:\, p(1) = 0 \,\}.$$

A. $\{\, 1 - 2x + x^2,\, 2 - 3x + x^2 \,\}$

B. $\{\, 1 - x \,\}$

C. $\{\, 1 - 2x + x^2 \,\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only B and C.
5. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),\, (1,1,0)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(0,0,0),\, (1,1,1),\, (1,1,0)\}) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\text{span}\left\{\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right\}\right) = 2$ in M_{22}

Select from the following:

1. All of A, B and C.
2. Only A.
3. Only A and B.
4. Only A and C.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

A. $\{\, \begin{bmatrix}-1 & 1 & 1 & -1\end{bmatrix} \,\}$

B. $\{\, \begin{bmatrix}1 & -1\end{bmatrix} \,\}$

C. $\{\, \begin{bmatrix}1 & -1\end{bmatrix},\, \begin{bmatrix}-1 & 1\end{bmatrix} \,\}$

Select from the following:

1. Only A.
2. Only B.

[TURN OVER]

3. Only C.
4. Only A and C.
5. None of the above.

(1.7) Which of the following sets are a basis for the column space of $\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

A. $\{ [1], [-1] \}$

B. $\{ [-1] \}$

C. $\{ \begin{bmatrix}1 & -1\end{bmatrix} \}$

D. $\{ \begin{bmatrix}1 & -1\end{bmatrix}, \begin{bmatrix}-1 & 1\end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only D.
5. None of the above.

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

1. $\text{rank}(A) = 0$, $\text{nullity}(A) = 4$.
2. $\text{rank}(A) = 1$, $\text{nullity}(A) = 3$.
3. $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.
4. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
5. None of the above.

[16]

QUESTION 2

Consider the vector space P_3.

(2.1) Show that (12)

$$\langle p(x), q(x) \rangle := p_0q_0 + 2p_1q_1 + 2p_2q_2 + p_3q_3,$$

where

$$p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 \quad \text{and} \quad q(x) = q_0 + q_1x + q_2x^2 + q_3x^3,$$

is an inner product on P_3.

(2.2) Are the vectors (6)

$$1 + x^2 + x^3,\ -1 - x^2 + x^3,\ -1 + x - x^2 + x^3$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of P_3: (12)

$$\{ 1 + x^2 + x^3,\ -1 - x^2 + x^3,\ -1 + x - x^2 + x^3 \}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

[30]

[TURN OVER]

QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 2 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda^2(\lambda - 3) = 0.$$

(3.3) Find bases for the eigenspaces of A. (18)

(3.4) Is A diagonalizable? Motivate your answer. (2)

(3.5) Let B be an $n \times n$ matrix and B^T be the transpose of B. Prove that: (5)

B is diagonalizable if and only if B^T is diagonalizable.

Hint: recall that $(P^{-1})^T = (P^T)^{-1}$ for any invertible matrix P.

[30]

QUESTION 4

Let $T : M_{22} \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b) + (c, d)$$

where $a, b, c, d \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (10)

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

in M_{22}, and the basis

$$B' = \{ (1, 1), (1, -1) \}$$

in $\mathbb{R}^2$, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = \mathbb{R}^2$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

[24]

TOTAL MARKS: [100]

UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611 (494246) October/November 2014

LINEAR ALGEBRA

Duration : 2 Hours 100 Marks

EXAMINERS :
FIRST : PROF Y HARDY
SECOND : PROF JD BOTHA

Use of a non-programmable pocket calculator is permissible.

Partial/limited open book examination. Specified material as indicated on examination paper, permissible.

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue.

This examination question paper consists of 5 pages.

ANSWER ALL THE QUESTIONS.
ALL CALCULATIONS MUST BE SHOWN.
ANY LINEAR ALGEBRA TEXTBOOK IS PERMITTED.

QUESTION 1

This question is a **multiple choice** question and should be filled in on the multiple choice **answer sheet** (mark reading sheet).

(1.1) Consider the set (2)

$$X := \{\,(x, y) \,:\, x, y \in \mathbb{R}\,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, &\qquad k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx + k - 1, ky), \\ + : X \times X \to X, &\qquad \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha + 1, y + \beta). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $\mathbf{0} = (-1, 0)$
2. $\mathbf{0} = (1, 0)$
3. $\mathbf{0} = (0, 0)$
4. $\mathbf{0} = (0, 1)$
5. None of the above.

(1.2) Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ? (2)

[TURN OVER]

A. $\operatorname{span}\{(0,0)\}$

B. $\{(x, x+1) : x \in \mathbb{R}\}$

C. $\{(0, x) : x \in \mathbb{R}, x \geq -1\}$

D. $\{(x, y+1) : x, y \in \mathbb{R}\}$

Select from the following:

1. Only A.
2. Only A and D.
3. Only C.
4. Only C and D.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. $\operatorname{span}\{1+x, 1-x\}$ in P_1

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\}$ in M_{22}

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ in M_{22}

D. $\{1+x, 1-x\}$ in P_2

Select from the following:

1. Only A.
2. Only B and C.
3. Only B and D.
4. Only D.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \{ A \in M_{22} : \operatorname{tr} A = 0 \}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.

3. Only A and B.
4. Only C and D.
5. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),(1,1,0)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(-1,-1,-1),(1,1,1)\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,2,1)\}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B.
4. Only A and C.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$? (2)

A. $\{ \begin{bmatrix} -1 & 1 \end{bmatrix} \}$

B. $\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \}$

D. $\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only D.
5. None of the above.

(1.7) Which of the following sets are a basis for the column space of $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T, \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T \right\}$

C. $\{ \begin{bmatrix} 1 & -1 \end{bmatrix} \}$

D. $\left\{ \begin{bmatrix} 3 & 0 & -3 \end{bmatrix}^T \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only B and D.
4. Only C.
5. None of the above.

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 1$, $\text{nullity}(A) = 1$.
2. $\text{rank}(A) = 1$, $\text{nullity}(A) = 0$.
3. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.
4. $\text{rank}(A) = 2$, $\text{nullity}(A) = 0$.
5. None of the above.

[16]

QUESTION 2

Consider the vector space $\mathbb{R}^4$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + 2x_2y_2 + 2x_3y_3 + x_4y_4, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$$

is an inner product on $\mathbb{R}^4$.

(2.2) Are the vectors (6)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^4$: (12)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

[30]

[TURN OVER]

QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 5 & -3 & 1 \\ 2 & -2 & 2 \\ 1 & -3 & 5 \end{bmatrix}.$$

(3.1) Determine the rank of A (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 4)^2 = 0.$$

(3.3) Find bases for the eigenspaces of A. (18)

(3.4) Is A diagonalizable? Motivate your answer. (2)

(3.5) Is the matrix $A + A^2$ diagonalizable? Motivate your answer. (2)

[30]

QUESTION 4

Let $T : \mathbb{R}^2 \to M_{22}$ be defined by

$$T(a, b) = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$$

where $a, b \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (10)

$$B = \{ (1, 1), (1, -1) \}$$

in $\mathbb{R}^2$, and the basis

$$B' = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

in M_{22}, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

[24]

TOTAL MARKS: [100]

UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611 (481090) May/June 2014

LINEAR ALGEBRA

Duration : 2 Hours 100 Marks

EXAMINERS :
FIRST : PROF Y HARDY
SECOND : PROF JD BOTHA

Use of a non-programmable pocket calculator is permissible.

Partial/limited open book examination. Specified material as indicated on examination paper, permissible.

This examination question paper consists of 5 pages.

ANSWER ALL THE QUESTIONS.
ALL CALCULATIONS MUST BE SHOWN.
ANY LINEAR ALGEBRA TEXTBOOK IS PERMITTED.

QUESTION 1

This question is a **multiple choice** question and should be filled in on the multiple choice **answer sheet** (mark reading sheet).

(1.1) Consider the set (2)

$$X := \{\, (x, y) \,:\, x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky),$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta).$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $-(0, 0) = (1, 0)$
2. $-(0, 0) = (1, 1)$
3. $-(0, 0) = (0, 1)$
4. $-(0, 0) = (2, 0)$
5. None of the above.

(1.2) Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ? (2)

[TURN OVER]

A. $\operatorname{span}\{(2,3)\}$

B. $\{(x,1) : x \in \mathbb{R}\}$

C. $\{(0,x) : x \in \mathbb{R},\, x \geq 0\}$

D. $\{(0,x-1) : x \in \mathbb{R}\}$

Select from the following:

1. Only A.
2. Only A and D.
3. Only C.
4. Only C and D.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. $\operatorname{span}\{(2,3)\}$ in $\mathbb{R}^2$

B. $\{(1,1),\, (-1,1)\}$ in $\mathbb{R}^2$

C. $\{(2,4),\, (1,-1),\, (1,1)\}$ in $\mathbb{R}^2$

D. $\{1+x,\, 1-x\}$ in P_1

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B and D.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of P_2: (2)

$$X = \{\, p(x) \in P_2 : p(1) = 0 \,\}.$$

A. $\left\{1,\, x,\, x^2\right\}$

B. $\left\{1-x,\, 1-x^2\right\}$

C. $\left\{1,\, 1-x,\, 1-x^2\right\}$

D. $\left\{1-x,\, 1-x^2,\, 3-2x-x^2\right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only A and C.
4. A, B, C and D.
5. None of the above.

[TURN OVER]

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),(1,1,-1)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(0,0,0),(1,1,1)\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,1,-1)\}) = 2$ in $\mathbb{R}^3$

D. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,1,-1)\}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A and B.
3. Only A and C.
4. Only A and D.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \}$

B. $\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & -1 \end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only B.
3. Only A and B.
4. A, B and C.
5. None of the above.

(1.7) Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 2 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 \end{bmatrix}^T \right\}$

D. $\{ \begin{bmatrix} 0 & 3 & -3 \end{bmatrix} \}$

Select from the following:

1. Only D.
2. Only A, B and C.
3. Only A and B.
4. Only A and C.
5. None of the above.

[TURN OVER]

(1.8) Which of the following sets are a basis for the null space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} -1 & 3 & 3 \end{bmatrix}^T \right\}$

Select from the following:

1. Only B and C.
2. Only B.
3. Only C.
4. Only A.
5. None of the above.

[16]

QUESTION 2

Consider the vector space $\mathbb{R}^4$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$$

is an inner product on $\mathbb{R}^4$.

(2.2) Are the vectors (6)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^4$: (12)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

[30]

[TURN OVER]

QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(3.1) Determine the rank of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 1)^2 = 0.$$

(3.3) Find bases for the eigenspaces of A. (18)

(3.4) Is A diagonalizable? Motivate your answer. (2)

(3.5) Is the matrix $A - I_3$ diagonalizable? Motivate your answer. (Here I_3 is the 3×3 identity matrix). (2)

[30]

QUESTION 4

Let $T : M_{22} \to P_2$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + \frac{b-c}{2}x + dx^2$$

where $a, b, c \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (12)

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, and the basis

$$B' = \{ 1 + x, 1 - x, x^2 \}$$

in P_2, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = P_2$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

[24]

TOTAL MARKS: [100]