UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

MAT3706

May/June 2017

ORDINARY DIFFERENTIAL EQUATIONS

Duration : 2 Hours

100 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT.

Closed book examination.

This paper consists of 4 pages

Instructions

- The notation is the same as in the Study Guide

Question 1.

Mark Distribution for this question : $10 + 10 = 20$ *marks*

(a) Consider the system

$$\begin{aligned} P_{11}(D)[x] + P_{12}(D)[y] + P_{13}(D)[z] &= f_1(t) \\ P_{21}(D)[x] + P_{22}(D)[y] + P_{23}(D)[z] &= f_2(t) \\ P_{31}(D)[x] + P_{32}(D)[y] + P_{33}(D)[z] &= f_3(t) \end{aligned} \tag{1}$$

of differential equations wherein each P_{ij} $(i, j = 1, 2, 3)$ are polynomials in a single variable, and f_1, f_2, f_3 are differentiable functions of t.

Suppose the system in (1) is reduced to the following triangular system:

$$\begin{aligned} P'_{11}(D)[x] &= g_1(t) \\ P'_{21}(D)[x] + P'_{22}(D)[y] &= g_2(t). \\ P'_{31}(D)[x] + P'_{32}(D)[y] + P'_{33}(D)[z] &= g_3(t) \end{aligned} \tag{2}$$

Explain clearly, if possible with an example, the meaning of reducing the system in (1) to a triangular form in (2), and how does it help in solving the system (1).

(b) Solve the following system of differential equations.

$$\begin{aligned} \ddot{x} + x + 4\ddot{y} - 4y &= 4e^t \\ \dot{x} - x + \dot{y} + 9y &= 0 \end{aligned} \tag{3}$$

Question 2.

Mark Distribution for this question . $10 + 10 = 20$ *marks*

(a) Given the system:

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y + a_{13}z \\ \dot{y} &= a_{21}x + a_{22}y + a_{23}z \\ \dot{z} &= a_{31}x + a_{32}y + a_{33}z \end{aligned}$$

where a_{ij} (for all $i, j = 1, 2, 3$) are fixed real numbers, let λ be an eigen-value of the coefficient matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with multiplicity 2.

Show that λ produces two linearly independent solutions of the system.

[TURN OVER]

(b) Find the general solution of the system:

$$\begin{aligned} \dot{x} &= y \\ y &= z \\ \dot{z} &= -2x - 5y - 4z \end{aligned}$$

Question 3.

Mark Distribution for this question : $10 + 10 = 20$ *marks*

(a) Prove : if $\boldsymbol{F}(t)$ is continuous and $\boldsymbol{\Phi}$ is a normalized fundamental matrix of $\boldsymbol{X} = \boldsymbol{AX}$ at $t_0 = 0$, then

$$\boldsymbol{X}_p(t) = \int_0^t \boldsymbol{\Phi}(t-s)\boldsymbol{F}(s)\,\mathrm{d}\,s$$

is a particular solution of $\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{F}$.

(b) Use the result in 3(a) to find a particular solution of :

$$\dot{\boldsymbol{X}} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \boldsymbol{X} + \begin{bmatrix} 1 \\ \cot t \end{bmatrix},$$

where $t_0 = \frac{\pi}{2}$

(**Recall :** $\int \operatorname{cosec} u d\, u = \ln|\operatorname{cosec} u + \cot u|$.)

Question 4.

Mark Distribution for this question : $5 + 5 + 5 + 5 = 20$ *marks.*

(a) If $[a, b] \xrightarrow{f} \mathbb{R}$ and $[a, b] \xrightarrow{g} \mathbb{R}$ are two real valued continuous functions with f differentiable and $\dot{f}(x) \leq f(x)g(x)$, $x \in [a, b]$ then show that for each $x \in [a, b]$:

$$f(x) \leq f(a)\mathrm{e}^{\int_a^x g(t)\mathrm{d}t}.$$

[**Hint**: Show that the function $h(t) = f(t)\mathrm{e}^{-\int_a^t g(x)\mathrm{d}x}$ is non-increasing on $[a, b]$.]

(b) State the Inequality of Gronwall.

(c) Show that the only non-negative contiuous function $[a, b] \xrightarrow{f} \mathbb{R}$ satisfying:

$$0 \leq f(t) \leq \int_a^t f(x)\mathrm{d}x, \quad \text{for } t \in [a, b]$$

is identically the zero function.

(d) Find the solution of the initial value problem on the interval $[a, b]$:

$$x = g(t)x, \quad x(a) = 0,$$

where $[a, b] \xrightarrow{g} \mathbb{R}$ is continuous.

Question 5.

Mark Distribution for this question : $4 + 16 = 20$ *marks.*

Consider the differential equation:

$$\frac{d^2y}{dt^2} + 4y = \sin 3t \tag{4}$$

(a) Construct the companion matrix and hence the companion system of (4).

(b) Solve the system completely
[**Recall**: $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \cos(\alpha - \beta)$]

TOTAL: [100]