Tutorial Letter 201/1/2018

Ordinary Differential Equations

MAT3706

Semester 1

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 01.

BAR CODE

Learn without limits.

UNISA university of south africa

SOLUTIONS TO ASSIGNMENT 02

Question 1 *Show that the system:*

$$(D-2)[x]+2D[y]=2-4\mathrm{e}^{2t}$$
$$(2D-3)[x]+(3D-1)[y]=0$$

is equivalent to both the following triangular systems:

$$\begin{cases}(D^2+D-2)[x]=2+20\mathrm{e}^{2t}\\ D[x]-2y=12\mathrm{e}^{2t}-6\end{cases} \quad \text{and} \quad \begin{cases}(D^2+D-2)[y]=-6-4\mathrm{e}^{2t}\\ x-(D+1)[y]=8\mathrm{e}^{2t}-4\end{cases}.$$

Given the three systems above, separately, state (with complete justifications) your strategy in solving them completely

Solution

(i) The system in operator notation is :

(1) $$(D-2)[x]+2D[y]=2-4\mathrm{e}^{2t}$$
(2) $$(2D-3)[x]+(3D-1)[y]=0$$

Multiplying the equation (2) *by* 2*yields :*

(3) $$(4D-6)[x]+(6D-2)[y]=0, ✓✓$$

which when subtracted from $3\times$ (1) *yields :*

(4) $$D[x]-2y=-6+12\mathrm{e}^{2t}.✓✓$$

from (4) *we have* $y=\frac{1}{2}D[x]+3-6\mathrm{e}^{2t}$✓ *Substituting it in the first equation* (1) *yields :*

(5) $$(D-2)[x]+2D\left(\tfrac{1}{2}D[x]+3-6\mathrm{e}^{2t}\right)=2-4\mathrm{e}^{2t}\Rightarrow(D^2+D-2)[x]=2+20e^{2t}✓$$

from (4) *and* (5) *we have the required triangular system :*

(6) $$(D^2+D-2)[x]=2+20\mathrm{e}^{2t}$$
(7) $$D[x]-2y=-6+12\mathrm{e}^{2t}.✓✓$$

(8 marks)

(ii) The given system is :

$$(1) \qquad (D-2)[x]+2D[y]=2-4\mathrm{e}^{2t}$$
$$(2) \qquad (2D-3)[x]+(3D-1)[y]=0$$

in which (1) *is retained* ✓ *and the second is replaced by* (2) -2(1) ✓ *:*

$$(3) \qquad x(t)-(D+1)[y]=-4+8e^{2t}, \quad ✓$$

which yields :

$$(4) \qquad x(t)=(D+1)[y]-4+8e^{2t}. \quad ✓$$

Substituting (4) *into* (1) *yields :*

$$(5) \quad (D-2)\left((D+1)[y]-4+8e^{2t}\right)+2D[y]=2-4e^{2t} \Rightarrow (D^2+D-2)[y]=-6-4e^{2t}, \quad ✓✓$$

so in view of (3) *and* (5) *the required triangular system is :*

$$(6) \qquad (D^2+D-2)[y]=-6-4e^{2t}$$
$$(7) \qquad x-(D+1)[y]=-4+8e^{2t} \quad ✓✓$$

(8 marks)

From (1) *and* (2) *we have* $\Delta=\begin{vmatrix} D-2 & 2D \\ 2D-3 & -3D-1 \end{vmatrix}=(-\frac{1}{2}D^2+D)(D-2-D)=2-D-D^2\neq 0$✓, *so that the system in* **not** *degenerate and hence has a unique solution.*✓ *In same procedure we can see* $\Delta\neq 0$ *for both triangular systems.*✓
For solving the triangular system it is easier than the original system. From we can find $x(t)$ *then substitute it into the we will obtain* $y(t)$ *(The Calculation has 4 marks).*
In same procedure we can find $y(t)$ *from then substitute it into the we will obtain* $x(t)$ *(The Calculation has 4 marks).*

(10 marks)
[26 marks]

Question 2 *Solve the system:*

$$(D^2+1)[x_1]-2D[x_2]=2t,$$
$$(2D-1)[x_1]-(2-D)[x_2]=5,$$

by using the elimination method (operator method).

Solution

Given the system:

$$(1) \qquad (D^2+1)[x_1]-2D[x_2]=2t$$
$$(2) \qquad (2D-1)[x_1]-(2-D)[x_2]=5$$

Eliminating $[x_2]$ first, we calculate $(2-D)\times$ **(1)** $-(2D)[$**(2)**$]$✓✓to get:

(3) $$(-D^3-2D^2+D+2)[x_1]=4t-2;✓$$

From **(3)** we solve for x_1. The auxiliary equation is $m^3+2m^2-m-2=(m-1)(m+1)(m+2)=0\Leftrightarrow m=\pm 1,-2,$✓yielding the complimentary function:

(4) $$x_{1C.F.}(t)=c_1e^{-2t}+c_2e^{-t}+c_3e^t,✓✓✓$$

where c_1, c_2 and c_3 are arbitrary constants.
The particular integral is given by:

$$\begin{aligned}
x_{1P.I.}(t)&=\frac{4t-2}{(D-1)(D+1)(D+2)}\\
&=\frac{1}{(D+2)(D-1)}\Big[1-D+D^2-D^3+\dots\Big](4t-2)\\
&=\frac{1}{(D+2)(D-1)}(4t-2-4)=\frac{1}{(D+2)(D-1)}(4t-6)\\
&=\frac{1}{D+2}\Big[1+D+D^2+\dots\Big](4t-6)\\
&=\frac{1}{D+2}(4t-6+4)=\frac{-1}{D+2}(4t-2)=\frac{-\frac{1}{2}}{1+\frac{D}{2}}(4t-2)\\
&=\frac{1}{2}\Big[1-\frac{D}{2}+\frac{D^2}{4}-\dots\Big](4t-2)\\
&=\frac{1}{2}(4t-2-2)=\frac{1}{2}(4t-4)=2t-2,
\end{aligned}$$

i.e.,

(5) $$x_{1P.I.}(t)=2t-2.✓✓$$

Hence $x_1=x_{1C.F.}+x_{1P.I.}$ yields the general solution for x_1 to be:

(6) $$x_1(t)=c_1e^{-2t}+c_2e^{-t}+c_3e^t+2t-2.✓$$

Now we find $x_2(t)$. From **(1)** we have ;

(7) $$D[x_2]=\frac{1}{2}(D^2+1)[x_1]-t✓$$

Now substitute **(6)** in the above equation **(7)** yields

(8) $D[x_2]=\frac{1}{2}(D^2+1)[x_1]-t=\frac{1}{2}(D^2+1)[c_1e^{-2t}+c_2e^{-t}+c_3e^t+2t-2]-t=\frac{5}{2}c_1e^{-2t}+c_2e^{-t}+c_3e^t-1.$✓

So by integrating of the above equation we can obtain $x_2(t)=-\frac{5}{4}c_1e^{-2t}-c_2e^{-t}+c_3e^t-t$✓✓Thus the solution is:

$$x_1(t)=c_1e^{-2t}+c_2e^{-t}+c_3e^t+2t-2,$$

and

$$x_2(t)=-\frac{5}{4}c_1e^{-2t}-c_2e^{-t}+c_3e^t-t.$$

[14 **marks]**

Question 3 *Consider the system of linear differential equations:*

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}.$$

(i) Suppose λ is an eigen value of $\mathbf{A}$ of multiplicity $k > 1$. Define a root vector corresponding to the eigen value λ.

(ii) Show that each eigen value λ of $\mathbf{A}$ of multiplicity k has k linearly independent root vectors.

(iii) Suppose that λ is an eigen value of $\mathbf{A}$ of multiplicity k and $\mathbf{U}_{\lambda,1}$, $\mathbf{U}_{\lambda,2}$, $\ldots$, $\mathbf{U}_{\lambda,k}$ are root vectors corresponding to λ of orders $1, 2, \ldots, k$, respectively. Further, let:

$$\mathbf{X}_i(t) = \mathrm{e}^{\lambda t}\big[\mathbf{U}_{\lambda,i} + t\mathbf{U}_{\lambda,i-1} + \frac{t^2}{2!}\mathbf{U}_{\lambda,i-2} + \ldots \frac{t^{i-1}}{(i-1)!}\mathbf{U}_{\lambda,1}\big], \text{ for } i = 1, 2, \ldots k.$$

Show that $\left\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_k(t)\right\}$ is a set of linearly independent solutions of the system of linear differential equations and every solution of the system is a linear combination of the solutions in this set.

Solution

(i) **Definitions 3.1 or 3.2** (5 marks)

(ii) **Lemma 3.10** (8 marks)

(iii) It is a combination of **Theorem 3.8 and Lemma 3.11**. (12 marks)

[25 marks]

Question 4 *Solve the system:*

$$\dot{\mathbf{X}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Why is there a unique solution to the above system?

Solution

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 2 & 1 & -2-\lambda \end{vmatrix} = -\lambda^3 - 2\lambda^2 + \lambda + 2 = (1-\lambda^2)(\lambda+2) = 0, \text{i.e., } \lambda = -1 \text{ or } \lambda = 1 \text{ or } \lambda = -2. ✓✓$$

For $\lambda = -1$:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, ✓$$

i.e., $\mathbf{X}_1(t) = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t}$ are the solutions corresponding to $\lambda = -1$.✓

For $\lambda = 1$:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \checkmark$$

i.e., $\mathbf{X}_2(t) = c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t$ are the solutions corresponding to $\lambda = 1$.✓

For $\lambda = -2$:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \checkmark$$

i.e., $\mathbf{X}_3(t) = c_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2t}$ are the solutions corresponding to $\lambda = -2$.✓

Hence the solutions are:

$$\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t) + \mathbf{X}_3(t) = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2t}, \checkmark$$

where c_1, c_2, c_3 are arbitrary constants.✓

Also in view of the initial condition at $t = 0$ we have:

$$\mathbf{X}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \checkmark \Rightarrow \begin{cases} c_1 + c_2 + c_3 = 1, \\ -c_1 + c_2 - 2c_3 = 0, \\ c_1 + c_2 + 4c_3 = 1 \end{cases} \checkmark \Rightarrow \begin{cases} c_1 = \frac{1}{2} = c_2, \\ c_3 = 0, \end{cases} \checkmark$$

yielding:

$$\mathbf{X}(t) = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} \checkmark$$

According to theorem 2.29, We have 3 real different eigenvalues then $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, $\mathbf{X}(0) = \mathbf{X}_0$ has a unique solution.✓ [15 marks]

Question 5 *Solve the system:*

$$\dot{\mathbf{X}} = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix} \mathbf{X}$$

completely.

Find the unique solution that passes through the point $(11, 3, -7)$ *when* $t = 0$.

Solution

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = (1-\lambda)(\lambda-2)^2 = 0, \checkmark\checkmark$$ i.e., $\lambda = 1$ or $\lambda = 2$ with multiplicity 2.✓✓✓

For $\lambda = 1$:

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \checkmark\checkmark$$

i.e., $\mathbf{X}_1(t) = c_1 \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} e^t$ are the solutions corresponding to $\lambda = 1$.✓

For $\lambda = 2$:

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \checkmark\checkmark$$

i.e., $\mathbf{X}_2(t) = c_2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} e^{2t}$ are one set of solutions corresponding to $\lambda = 2$.✓

To find a second solution, we solve:

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -\frac{3}{2} \end{bmatrix}, \checkmark$$

i.e., $\mathbf{X}_3(t) = c_3 \left[\begin{bmatrix} 3 \\ 1 \\ -\frac{3}{2} \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right] e^{2t}$ are another set of solutions corresponding to $\lambda = 2$.✓✓

Hence the solutions are:

$$\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t) + \mathbf{X}_3(t) = c_1 \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} e^{2t} + c_3 \left[\begin{bmatrix} 3 \\ 1 \\ -\frac{3}{2} \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \right] e^{2t}, \checkmark$$

where c_1, c_2, c_3 are arbitrary constants.✓

Also:

$$\mathbf{X}(0) = \begin{bmatrix} 11 \\ 3 \\ -7 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ -\frac{3}{2} \end{bmatrix} \checkmark \Rightarrow \begin{cases} 4c_1 + 3c_2 + 3c_3 = 11, \\ c_1 + c_2 + c_3 = 3, \\ -3c_1 - 2c_2 - \frac{3}{2}c_3 = -7 \end{cases} \checkmark \Rightarrow \begin{cases} c_1 = 2 = c_3, \\ c_2 = -1, \end{cases} \checkmark\checkmark$$

yielding:

$$\mathbf{X}(t) = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + 3 \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right] e^{2t} = \begin{bmatrix} 8e^t + (3+6t)e^{2t} \\ 2e^t + (1+2t)e^{2t} \\ -6e^t - (1+4t)e^{2t} \end{bmatrix}. \checkmark\checkmark$$

[20 marks]