Tutorial Letter 202/1/2018

Ordinary Differential Equations

MAT3706

Semester 1

Department of Mathematical Sciences

This tutorial letter contains solutions
for assignment 02.

BAR CODE

SOLUTIONS TO ASSIGNMENT 02

Question 1 *Solve the following initial value problem*

$$\dot{\mathbf{X}} = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

Solution

Since the matrix is an Upper Triangular Matrix the eigenvalues are diagonal elements. So $\lambda = 2$ with multiplicity 3 ✓✓.

For $\lambda = 2$:

$$\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \checkmark$$

i.e., $\mathbf{X}_1(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$ are one set of solutions corresponding to $\lambda = 2$.✓

To find a second solution, we solve:

$$\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \checkmark\checkmark$$

i.e., $\mathbf{X}_2(t) = c_2 \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] e^{2t}$ are another set of solutions corresponding to $\lambda = 2$.✓✓To find a third solution, we solve:

$$\begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{5} \\ \frac{1}{5} \end{bmatrix}, \checkmark\checkmark$$

i.e., $\mathbf{X}_3(t) = c_3 \left[\begin{bmatrix} 0 \\ -\frac{1}{5} \\ \frac{1}{5} \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] e^{2t}$ are another set of solutions corresponding to $\lambda = 2$.✓✓Hence the solutions are:

$$\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t) + \mathbf{X}_3(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] e^{2t} + c_3 \left[\begin{bmatrix} 0 \\ -\frac{1}{5} \\ \frac{1}{5} \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] e^{2t}, \checkmark\checkmark$$

where c_1, c_2, c_3 are arbitrary constants.✓

Also:

$$\mathbf{X}(0)=\begin{bmatrix}2\\-3\\1\end{bmatrix}=c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\-\frac{1}{5}\\\frac{1}{5}\end{bmatrix}\checkmark\Rightarrow\begin{cases}c_1=2,\\c_2-\frac{c_3}{5}=-3,\\\frac{c_3}{5}=1\end{cases},\checkmark\Rightarrow\begin{cases}c_1=2,\\c_3=5,\\c_2=-2\end{cases},\checkmark$$

yielding:

$$\mathbf{X}(t)=2\begin{bmatrix}1\\0\\0\end{bmatrix}e^{2t}-2\left[\begin{bmatrix}0\\1\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\0\end{bmatrix}\right]e^{2t}+5\left[\begin{bmatrix}0\\-\frac{1}{5}\\\frac{1}{5}\end{bmatrix}+t\begin{bmatrix}0\\1\\0\end{bmatrix}+\frac{t^2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}\right]e^{2t},=\begin{bmatrix}2-2t+\frac{5}{2}t^2\\-3+t\\1\end{bmatrix}e^{2t}.\checkmark\checkmark$$

[20 marks]

Question 2 *If ϕ is a normalized fundamental matrix $\dot{\mathbf{X}}=\mathbf{AX}$ at $t_0=0$ and if $\mathbf{F}$ is continuous in t, then*

$$\mathbf{X}_p(t)=\int_0^t\phi(t-s)\mathbf{F}(s)\,ds,$$

is a particular solution of $\dot{\mathbf{X}}=\mathbf{AX}+\mathbf{F}$. Use the result above to find a general solution of

$$\dot{\mathbf{X}}=\begin{pmatrix}2&1\\-4&2\end{pmatrix}\mathbf{X}+\begin{pmatrix}t\,e^{2t}\\-e^{2t}\end{pmatrix}.$$

Solution

The characteristic function of the matrix is $c(\lambda)=\det\begin{bmatrix}2-\lambda&1\\-4&2-\lambda\end{bmatrix}=(2-\lambda)^2+4=\lambda^2-4\lambda+8\checkmark$, so that the eigenvalues are $\lambda=2\pm2i\checkmark\checkmark$;
For $\lambda=2+2i$, the associated eigenvector is obtained from the solution of :

$$\begin{pmatrix}-2i&1\\-4&-2i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\checkmark\Rightarrow\begin{cases}-2iv_1+v_2=0\quad(1)\\-4v_1-2iv_2=0\quad(2)\end{cases}\checkmark$$

The equation (2) is equivalent equation (1) $[-2i\times(1)=(2)]$. So from (1) we have $-2iv_1+v_2=0\to v_2=2iv_1$. We take $v_1=1$ so that $V=\begin{bmatrix}1\\2i\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+i\begin{bmatrix}0\\2\end{bmatrix}.\checkmark$is an eigenvector. So in view of **Theorem 2.19 page 32 of S.G.** we have

$$\begin{cases}\boldsymbol{X}_1(t)=e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix}\cos 2t-\begin{pmatrix}0\\2\end{pmatrix}\sin t\right)&=e^{2t}\begin{pmatrix}\cos 2t\\-2\sin 2t\end{pmatrix}\\\boldsymbol{X}_2(t)=e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix}\sin 2t+\begin{pmatrix}0\\2\end{pmatrix}\cos 2t\right)&=e^{2t}\begin{pmatrix}\sin 2t\\2\cos 2t\end{pmatrix}\end{cases},\checkmark$$

Two real solutions are $\begin{bmatrix}\cos 2t\\-2\sin 2t\end{bmatrix}e^{2t}\checkmark\checkmark$and $\begin{bmatrix}\sin 2t\\2\cos 2t\end{bmatrix}e^{2t},\checkmark\checkmark$while a fundamental matrix is :

$$\Phi(t)=\begin{bmatrix}\cos 2t&\sin 2t\\-2\sin 2t&2\cos 2t\end{bmatrix}e^{2t},\checkmark\checkmark$$

with $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \checkmark \Phi^{-1}(0) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \checkmark$
Normalised fundamental matrix is :

$$\Psi(t) = \Phi(t)\Phi^{-1}(0) = \frac{e^{2t}}{2}\begin{bmatrix} 2\cos 2t & \sin 2t \\ -4\sin 2t & 2\cos 2t \end{bmatrix}. \checkmark\checkmark$$

Then :

$$\begin{aligned}
X_p(t) &= \int_0^t \Psi(t-s)F(s)\mathrm{d}s \checkmark \\
&= \int_0^t \frac{e^{2(t-s)}}{2}\begin{bmatrix} 2\cos 2(t-s) & \sin 2(t-s) \\ -4\sin 2(t-s) & 2\cos 2(t-s) \end{bmatrix}\begin{bmatrix} se^{2s} \\ -e^{2s} \end{bmatrix}\mathrm{d}s \checkmark\checkmark \\
&= \int_0^t \frac{e^{2t}}{2}\begin{bmatrix} 2s\cos 2(t-s) - \sin 2(t-s) \\ -4s\sin 2(t-s) - 2\cos 2(t-s) \end{bmatrix}\mathrm{d}s \checkmark \\
&= \begin{bmatrix} \frac{1}{2}se^{2t}\sin(2(s-t))\Big|_0^t \\ s\left(-e^{2t}\right)\cos(2(s-t))\Big|_0^t \end{bmatrix} \checkmark \\
&= \begin{bmatrix} 0 \\ -te^{2t} \end{bmatrix}. \checkmark\checkmark
\end{aligned}$$

Hence the general solution is $X(t) = c_1\boldsymbol{X}_1(t) + c_2\boldsymbol{X}_2(t) + X_p(t) = c_1\begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix}e^{2t} + c_2\begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}e^{2t} + \begin{bmatrix} 0 \\ -te^{2t} \end{bmatrix}$ where c_1, c_2 arbitrary constant $\checkmark$

[25 marks]

Question 3 *Use Corollary 4.39 with $t_0 = 0$ of the study guide to solve the inhomogeneous initial value problem*

$$\dot{\mathbf{X}} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Solution

The characteristic function of the matrix is $c(\lambda) = \det\begin{bmatrix} 0-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix} = -\lambda(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2$, so that the eigenvalues are $\lambda = 1 \checkmark$ and $\lambda = 2 \checkmark$;
For $\lambda = 1$, the associated eigenvector is obtained from the solution of :

$$\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -v_1 + 2v_2 = 0 & (1) \\ -v_1 + 2v_2 = 0 & (2) \end{cases}$$

The equation (2) is equivalent equation (1). From (1) we have $-v_1 + 2v_2 = 0 \to v_1 = 2v_2$. We take $v_2 = 1$ so that $V = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. is an eigenvector. i.e., $\mathbf{X}_1(t) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^t$ are the solutions corresponding to

$\lambda = 1.$ ✓
For $\lambda = 2$:

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \checkmark$$

i.e., $\mathbf{X}_2(t) = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$ are the solutions corresponding to $\lambda = 2.$ ✓ a fundamental matrix is :

$$\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} e^{2t}, \checkmark$$

with $\Phi(0) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, ✓ $\Phi^{-1}(0) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. ✓
Normalised fundamental matrix is :

$$\Psi(t) = \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} 2e^t - e^{2t} & -2e^t + 2e^{2t} \\ e^t - e^{2t} & -e^t + 2e^{2t} \end{bmatrix}. \checkmark\checkmark$$

Then from (4.39) a particular solution is :

$$\begin{aligned}
\boldsymbol{X}_p(t) &= \int_0^t \Psi(t-s)F(s)\mathrm{d}s \checkmark \\
&= \int_0^t \begin{bmatrix} 2e^{t-s} - e^{2(t-s)} & -2e^{t-s} + 2e^{2(t-s)} \\ e^{t-s} - e^{2(t-s)} & -e^{t-s} + 2e^{2(t-s)} \end{bmatrix} \begin{bmatrix} 2 \\ e^{-3s} \end{bmatrix} \mathrm{d}s \checkmark \\
&= \int_0^t \begin{bmatrix} -2e^{t-4s} + 4e^{t-s} - 2e^{2(t-s)} + 2e^{2(t-s)-3s} \\ -e^{t-4s} + 2e^{t-s} - 2e^{2(t-s)} + 2e^{2(t-s)-3s} \end{bmatrix} \mathrm{d}s \checkmark \\
&= \begin{bmatrix} -\frac{1}{10}e^{t-5s}\left(-10e^{3s+t} - 5e^s + 40e^{4s} + 4e^t\right)\Big|_0^t \\ -\frac{1}{20}e^{t-5s}\left(-20e^{3s+t} - 5e^s + 40e^{4s} + 8e^t\right)\Big|_0^t \end{bmatrix} \checkmark \\
&= \begin{bmatrix} -3 + \frac{e^{-3t}}{10} + \frac{7e^t}{2} - \frac{3e^{2t}}{5} \\ -1 - \frac{3e^{-3t}}{20} + \frac{7e^t}{4} - \frac{3e^{2t}}{5} \end{bmatrix} \checkmark
\end{aligned}$$

Hence the general solution is
$X(t) = c_1\boldsymbol{X}_1(t) + c_2\boldsymbol{X}_2(t) + \boldsymbol{X}_p(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -3 + \frac{e^{-3t}}{10} + \frac{7e^t}{2} - \frac{3e^{2t}}{5} \\ -1 - \frac{3e^{-3t}}{20} + \frac{7e^t}{4} - \frac{3e^{2t}}{5} \end{bmatrix}$ where c_1, c_2 arbitrary constant ✓.
Also:

$$\mathbf{X}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark \Rightarrow \begin{cases} 2c_1 + c_2 = 2, \\ c_1 + c_2 = 4, \end{cases}, \checkmark \Rightarrow c_1 = -2, c_2 = 6, \checkmark$$

yielding:

$$\mathbf{X}(t) = -2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 + \frac{e^{-3t}}{10} + \frac{7e^t}{2} - \frac{3e^{2t}}{5} \\ -1 - \frac{3e^{-3t}}{20} + \frac{7e^t}{4} - \frac{3e^{2t}}{5} \end{bmatrix} = \begin{bmatrix} \frac{e^{-3t}}{10} - \frac{e^t}{2} + \frac{27e^{2t}}{5} - 3 \\ -1 - \frac{3e^{-3t}}{20} - \frac{e^t}{4} + \frac{27e^{2t}}{5} \end{bmatrix}. \checkmark$$

[20 marks]

Question 4 *Write the companion system for the equations given below.*

$$a)\ y^{(3)} - y\sin t = f(t), \qquad b)\ \ddot{y} - y = 0.$$

Solution

(a) From $y^{(3)} - y\sin t = f(t)$ we have $y^{(3)} = y\sin t + f(t)$✓. Let $\mathbf{X} = \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \end{bmatrix}$ then $\dot{\mathbf{X}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix}$✓. So that

$$\dot{\mathbf{X}} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin t & 0 & 0 \end{pmatrix} \begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f(t) \end{bmatrix} ✓✓✓$$

(b) From $\ddot{y} - y = 0$ we have $\ddot{y} = y$✓. Let $\mathbf{X} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$ then $\dot{\mathbf{X}} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix}$✓. So that

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} ✓✓✓$$

[10 marks]

Question 5 *Use the power series method to find a series solution for*

$$\dot{\mathbf{X}} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{X}, \quad \mathbf{X} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Write the final answer in term of the appropriate exponential function.

Solution

Since all the entries are constants, the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is analytic at $t = 0$.✓

The sum $\sum_{k=0}^{\infty} A_k (t - t_0)^k$ can yield the constant matrix only if A_1, A_2, ... are all zero matrix. Hence $A_0 = A$ and $A_k = 0$ for $k \geq 1$.✓Furthermore :

$$X_n = \frac{1}{n} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} X_{n-1} = \frac{1}{n!} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}^n X_0, \quad \text{for } n = 1, 2, \ldots. ✓✓$$

Now $X(t) = X_0 + \sum_{k=1}^{\infty} \frac{1}{k!} A^k X_0 t^k$,✓where we note from :

$$AX_0 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, ✓$$

$$A^2X_0 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}, ✓$$

$$A^3X_0 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 8 \end{bmatrix}, ✓$$

yielding the pattern :

$$A^k X_0 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k + 1 \\ 2^k \\ 2^k \end{bmatrix}, \text{ for } k = 0, 1, 2, \ldots. \quad \checkmark\checkmark$$

Hence :

$$X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 2^k + 1 \\ 2^k \\ 2^k \end{bmatrix}; \quad \checkmark\checkmark$$

but :

$$\sum_{k=0}^{\infty} \frac{2^k + 1}{k!} t^k = \sum_{k=0}^{\infty} \left(\frac{(2t)^k}{k!} + \frac{t^k}{k!} \right) = e^{2t} + e^t, \checkmark$$

yielding :

$$X(t) = \begin{bmatrix} e^{2t} + e^t \\ e^{2t} \\ e^{2t} \end{bmatrix}. \quad \checkmark\checkmark$$

[15 marks]

Question 6 *Classify the critical points of the plane autonomous system corresponding to the second order non-linear differential equation:*

$$\ddot{x} + \nu(x^2 - 1)\dot{x} + x = 0.$$

Solution

Let $y = \dot{x}$ and then $\dot{y} = -x - \nu(x^2 - 1)y.$✓✓
Observe $\dot{x} = 0 \Rightarrow y = 0$ and $\dot{x} = \ddot{x} = 0 \Rightarrow x = 0$, yielding the critical points $(0, 0)$ ✓.
Then :

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 2\nu xy & -\nu(x^2 - 1) \end{bmatrix} \checkmark \Rightarrow J(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & \nu \end{bmatrix} \checkmark$$

Since at $(0, 0)$ $\tau = \nu, \Delta = 1$✓,

- $\tau = \nu \begin{cases} \nu > 0 \text{unstable}, \\ \nu < 0 \text{stable}, \end{cases}$,✓

- $\tau^2 - 4\Delta \begin{cases} < 0 & \text{Spiral} \Rightarrow \nu < 0 - 2 < \nu < 2, \\ > 0 & \text{Node} \Rightarrow \begin{cases} \nu < -2 \\ \nu > 2 \end{cases} \end{cases}$ ✓

$\nu > 2$:unstable node.
$0 < \nu < 2$:unstable spiral.✓
$-2 < \nu < 0$:Stable spiral.
$\nu < -2$:Stable node.
$\nu = 2$: degenerate unstable node.✓
$\nu = -2$: degenerate stable node.

[10 marks]