Tutorial Letter 201/2/2018

Ordinary Differential Equations

MAT3706

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions
for assignment 01.

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SOLUTIONS TO ASSIGNMENT 02

Question 1 *Cnsider the following system of differential equations:*

$$\mathbf{P}_{11}(D)[x] + \mathbf{P}_{12}(D)[y] = f_1(t),$$
$$\mathbf{P}_{21}(D)[x] + \mathbf{P}_{22}(D)[y] = f_2(t).$$

(i) *How do we determine the correct number of arbitrary constants in a general solution of the above system.* *(5 marks)*

(ii) *Explain briefly the difference between the operator method and the method of triangularization when used for solving the above system.* *(15 marks)*

Solution

(i) $\Delta = \begin{vmatrix} \mathbf{P}_{11}(D) & \mathbf{P}_{12}(D) \\ \mathbf{P}_{21}(D) & \mathbf{P}_{22}(D) \end{vmatrix}$ the number of arbitrary constant is equal to the degree of Δ.

(5 marks)

(ii) When using the operator method, either x or y is immediately eliminated by either using the operations

$$\mathbf{P}_{21}(D)[1] - \mathbf{P}_{11}(D)[2]$$

to eliminate x or

$$\mathbf{P}_{22}(D)[1] - \mathbf{P}_{12}(D)[2]$$

to eliminate y yielding nly in x or y which is then solved in the usual way.
When using the method of triangularization, we consider all the polynomials $\mathbf{P}(D)$ and retain the equation in which the order of $\mathbf{P}(D)$ is the lowest, say for example eq.(2). Instead of eliminating x or y all together in one step, we reduce in each step the order of the coefficient polynomial $\mathbf{P}_{11}(D)$ of x (for example) by one until the coefficient polynomial of s is a constant.
This equation is then used to find an expression of x in terms of y which is then substituted into the 2nd equation to find an equation only in y. This leads to the triangular system which consists of one equation in y and another in x and y where the order of the coefficient polynomial of x has been reduced to a constant. Becuse of this special triangular form , this method of solution immediately leads to the correct number of constants in the general solution of the system of ODEs. (15 marks)

[20 **marks**]

Question 2 *Determine whether the system*

$$\dot{x} + x - \dot{y} - y = e^{\frac{t}{2}},$$
$$2\dot{x} - x - 2\dot{y} + y = 5e^{-t}$$

has no solution or infinitely solution. If it has no solution, explain why, else solve $y(t)$ in term of $x(t)$, where $x(t)$ is an arbitrary function of t.

Solution

[20 **marks]**

Question 3 *Solve the system:*

$$(D^2+1)[x_1]-2D[x_2]=2t,$$
$$(2D-1)[x_1]-(2-D)[x_2]=5,$$

by using the elimination method (operator method).

Solution

Given the system:

(1) $$(D^2+1)[x_1]-2D[x_2]=2t$$
(2) $$(2D-1)[x_1]-(2-D)[x_2]=5$$

Eliminating $[x_2]$ first, we calculate $(2-D)\times$ (1) $-(2D)[$(2)$]$✓✓to get:

(3) $$(-D^3-2D^2+D+2)[x_1]=4t-2;\checkmark$$

From (3) we solve for x_1. The auxiliary equation is $m^3+2m^2-m-2=(m-1)(m+1)(m+2)=0\Leftrightarrow m=\pm1,-2,$✓yielding the complimentary function:

(4) $$x_{1C.F.}(t)=c_1e^{-2t}+c_2e^{-t}+c_3e^t,\checkmark\checkmark\checkmark$$

where c_1, c_2 and c_3 are arbitrary constants.
The particular integral is given by: (**Note:** You can also use the method of undetermined coefficients to find particular solutions)

$$\begin{aligned}
x_{1P.I.}(t)&=\frac{4t-2}{(D-1)(D+1)(D+2)}\\
&=\frac{1}{(D+2)(D-1)}\Big[1-D+D^2-D^3+\dots\Big](4t-2)\\
&=\frac{1}{(D+2)(D-1)}(4t-2-4)=\frac{1}{(D+2)(D-1)}(4t-6)\\
&=\frac{1}{D+2}\Big[1+D+D^2+\dots\Big](4t-6)\\
&=\frac{1}{D+2}(4t-6+4)=\frac{-1}{D+2}(4t-2)=\frac{-\frac{1}{2}}{1+\frac{D}{2}}(4t-2)\\
&=\frac{1}{2}\Big[1-\frac{D}{2}+\frac{D^2}{4}-\dots\Big](4t-2)\\
&=\frac{1}{2}(4t-2-2)=\frac{1}{2}(4t-4)=2t-2,
\end{aligned}$$

i.e.,

(5) $$x_{1P.I.}(t)=2t-2.\checkmark\checkmark$$

Hence $x_1 = x_{1C.F.} + x_{1P.I.}$ yields the general solution for x_1 to be:

(6) $$x_1(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^t + 2t - 2.\checkmark$$

Now we find $x_2(t)$. From **(1)** we have ;

(7) $$D[x_2] = \frac{1}{2}(D^2 + 1)[x_1] - t\checkmark$$

Now substitute **(6)** in the above equation **(7)** yields

(8) $$D[x_2] = \frac{1}{2}(D^2+1)[x_1] - t = \frac{1}{2}(D^2+1)[c_1 e^{-2t} + c_2 e^{-t} + c_3 e^t + 2t - 2] - t = \frac{5}{2}c_1 e^{-2t} + c_2 e^{-t} + c_3 e^t - 1.\checkmark\checkmark$$

So by integrating of the above equation we can obtain $x_2(t) = -\frac{5}{4}c_1 e^{-2t} - c_2 e^{-t} + c_3 e^t - t\checkmark\checkmark$ Thus the solution is:

$$x_1(t) = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^t + 2t - 2,$$

and

$$x_2(t) = -\frac{5}{4}c_1 e^{-2t} - c_2 e^{-t} + c_3 e^t - t.$$

[15 **marks]**

Question 4 *Consider the system of linear differential equations:*

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}.$$

(i) Suppose λ is an eigen value of $\mathbf{A}$ of multiplicity $k > 1$. Define a root vector corresponding to the eigen value λ.

(ii) Show that each eigen value λ of $\mathbf{A}$ of multiplicity k has k linearly independent root vectors.

(iii) Suppose that λ is an eigen value of $\mathbf{A}$ of multiplicity k and $\mathbf{U}_{\lambda,1}$, $\mathbf{U}_{\lambda,2}$, $\ldots$, $\mathbf{U}_{\lambda,k}$ are root vectors corresponding to λ of orders $1, 2, \ldots, k$, respectively. Further, let:

$$\mathbf{X}_i(t) = \mathrm{e}^{\lambda t}\big[\mathbf{U}_{\lambda,i} + t\mathbf{U}_{\lambda,i-1} + \frac{t^2}{2!}\mathbf{U}_{\lambda,i-2} + \ldots \frac{t^{i-1}}{(i-1)!}\mathbf{U}_{\lambda,1}\big],\ \textit{for } i = 1, 2, \ldots k.$$

Show that $\left\{\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_k(t)\right\}$ is a set of linearly independent solutions of the system of linear differential equations and every solution of the system is a linear combination of the solutions in this set.

Solution

(i) **Definitions 3.1 or 3.2** (5 marks)

(ii) **Lemma 3.10** (8 marks)

(iii) It is a combination of **Theorem 3.8 and Lemma 3.11**. (12 marks)

[25 marks]

Question 5 *Solve the system:*

$$\dot{\mathbf{X}} = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} \mathbf{X}$$

completely.

Find the unique solution that passes through the point $(4, 5, -9)$ *when* $t = 0$.

Solution

$\begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = -\lambda^3 + 12\lambda^2 - 21\lambda + 10 = 0 \Rightarrow \lambda^3 - 12\lambda^2 + 21\lambda - 10 = (\lambda - 1)(\lambda^2 - 11\lambda + 10) =$ 0,✓✓i.e., $\lambda = 10$ or $\lambda = 1$ with multiplicity 2.✓✓✓

For $\lambda = 10$:

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{0} \Longrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, ✓✓$$

i.e., $\mathbf{X}_1(t) = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{10t}$ are the solutions corresponding to $\lambda = 10$.✓

For $\lambda = 1$ we solve:

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0}, ✓$$

which is equivalent to the equation $4v_1 + 4v_2 + 2v_3 = 0$✓

Since we have only one equations with 3 unknowns, two of the unknowns can be chosen arbitrarily and we can solve the third unknown in term of two chosen ones.✓

We therefore choose (for example) to solve $v - 1$ in terms of $v - 2$ and v_3. Then $v_1 = -v_2 - \frac{v_3}{2}$.✓

Hence $\begin{bmatrix} -v_2 - \frac{v_3}{2} \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ ✓. Since $v - 2$ and v_3 can be chosen arbitrarily we chose $v - 2 = c_2$ and $v_3 = c_3$✓ so that two linearly independent solution corresponding to $\lambda = 1$ are $\mathbf{X}_2(t) = c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^t$ ✓and $\mathbf{X}_3(t) = c_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} e^t$ ✓where c_2 and $c_3$are arbitrarily constants.

Hence the solutions are:

$$\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t) + \mathbf{X}_3(t) = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{10t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} e^t, ✓$$

where c_1, c_2, c_3 are arbitrary constants.✓

Also:

$$\mathbf{X}(0) = \begin{bmatrix} 4 \\ 5 \\ -9 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \checkmark \Rightarrow \begin{cases} 2c_1 - c_2 - \frac{c_3}{2} = 4, \\ 2c_1 + c_2 = 5, \\ c_1 + c_3 = -9 \end{cases} \checkmark \Rightarrow \begin{cases} c_1 = 1, \\ c_3 = -10, \\ c_2 = 3, \end{cases} \checkmark$$

yielding:

$$\mathbf{X}(t) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{10t} + 3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^t - 10 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} e^t = \begin{bmatrix} 2e^{10t} + 8e^t \\ 2e^{10t} - 3e^t \\ e^{10t} - 10e^t \end{bmatrix} . \checkmark$$

[20 marks]