Tutorial Letter 202/2/2018

Ordinary Differential Equations

MAT3706

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 02.

The Solution of Question 2 of Assignment 1 is Added.

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SOLUTIONS TO ASSIGNMENT 02

Question 1 *(a) If* $\mathbf{F}(t)$ *is continuous and* $\mathbf{\Phi}$ *is a fundamental matrix of* $\dot{\mathbf{X}} = \mathbf{AX}$, *then show that a particular solution of the non-homogeneous system* $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{F}(t)$ *is given by* ***(8)***

$$\mathbf{X}_p(t) = \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\,\mathbf{F}(s)\,ds.$$

(b) Use the result in Question 1(a) *with* $t_0 = 0$ *to find a* general *real solution of* ***(12)***

$$\dot{\mathbf{X}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{X} + 2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{2t}.$$

Solution

(a) **Theorem 4.36 page 60 of SG.** (8 marks)

(b) The characteristic function of the matrix is $c(\lambda) = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0$✓, so that the eigenvalues are $\lambda = 1 \pm i$✓;

For $\lambda = 1 + i$, the associated eigenvector is obtained from the solution of :

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -iv_1 - v_2 = 0 & (1) \\ v_1 - iv_2 = 0 & (2) \end{cases} \checkmark$$

The equation (2) is equivalent equation (1) $[-i \times (1) = (2)]$. So from (1) we have $-iv_1 - v_2 = 0 \rightarrow v_2 = -iv_1$. We take $v_2 = 1$ so that $V = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.✓is an eigenvector. So in view of **Theorem 2.19 page 32 of S.G.** we have

$$\begin{cases} \boldsymbol{X}_1(t) = e^t\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right) & = e^t\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \\ \boldsymbol{X}_2(t) = e^t\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right) & = e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \end{cases}, \checkmark$$

Two real solutions are $\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^t$ and $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t$, while a fundamental matrix is :

$$\Phi(t) = \begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix} e^t, \checkmark$$

with $\Phi(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\Phi^{-1}(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Normalised fundamental matrix is :

$$\Phi^{-1}(t) = \Phi^{-1}(0)\Phi(-t)\Phi^{-1}(0) = e^{-t}\begin{bmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{bmatrix}.\checkmark$$

Then the particular solution is therefore (1(a)):

$$\begin{aligned}
X_p(t) &= \Phi(t)\int_0^t \Phi^{-1}(s)F(s)\mathrm{d}s\checkmark \\
&= \Phi(t)\int_0^t e^{-s}\begin{bmatrix} -\sin s & \cos s \\ \cos s & \sin s \end{bmatrix} 2\begin{bmatrix} \cos s \\ \sin s \end{bmatrix} e^{2s}\mathrm{d}s \\
&= 2\Phi(t)\int_0^t e^{s}\begin{bmatrix} -\cos s\sin s + \sin s\cos s \\ \sin^2 s + \cos^2 s \end{bmatrix}\mathrm{d}s = \Phi(t)\int_0^t e^{s}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathrm{d}s\checkmark \\
&= 2\Phi(t)\begin{bmatrix} 0\Big|_0^t \\ e^{s}\Big|_0^t \end{bmatrix}\checkmark \\
&= 2\Phi(t)\begin{bmatrix} 0 \\ e^{t}-1 \end{bmatrix} = 2\begin{bmatrix} e^{2t}\cos t - e^{t}\cos t \\ e^{2t}\sin t - e^{t}\sin t \end{bmatrix}.\checkmark
\end{aligned}$$

Hence the general solution is $X(t) = c_1\boldsymbol{X}_1(t) + c_2\boldsymbol{X}_2(t) + X_p(t) = c_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^{t} + c_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{t} + 2\begin{bmatrix} e^{2t}\cos t - e^{t}\cos t \\ e^{2t}\sin t - e^{t}\sin t \end{bmatrix}$ where c_1, c_2 arbitrary constant $\checkmark$

(12 marks)

[20 **marks**]

Question 2 *(i) Show that for any square matrix* $\mathbf{A} = (a_{ij})_{i,j=1}^{n}$, *if:* *(6 marks)*

$$||\mathbf{A}|| = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2},$$

then: $||A|| = 0 \Leftrightarrow \mathbf{A} = \mathbf{0}$, $||\lambda\mathbf{A}|| = |\lambda|||\mathbf{A}||$ *and* $||\mathbf{A}+\mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$.

(ii) Suppose $[a,b] \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{R}$ *are real valued continuous functions defined on the interval* $[a,b]$ *with* f *differentiable and* $\dot{f}(x) \leq f(x)g(x)$ *for all* $x \in (a,b)$.
Show that for each $x \in [a,b]$*:* *(8 marks)*

$$f(x) \leq f(a)\mathrm{e}^{h(x)}, \qquad \text{where } h(x) = \int_a^x g(t)\mathrm{d}t.$$

(iii) *Using the statement of* (ii), *or otherwise, show that for real valued continuous functions* $[a,b] \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{R}$ *of which* g *is always non-negative and:*

$$f(x) \leq K + \int_a^x f(t)g(t)\mathrm{d}t, \qquad \textit{for all } x \in [a,b],$$

for some real number K*, one always has for each* $t \in [a,b]$*:* *(3 marks)*

$$f(t) \leq K\mathrm{e}^{h(t)}, \qquad \textit{where } h(t) = \int_a^t f(x)\mathrm{d}x.$$

(iv) *Show that for any solution* $\mathbf{X}(t)$ *of the system* $\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t)$*, for any* s *and* t*:* *(3 marks)*

$$||\mathbf{X}(t)|| \leq ||\mathbf{X}(s)||\ \mathrm{e}^{||\mathbf{A}||\,||t-s||}.$$

[20 marks]

Solution

(i) **This is Standard check of the properties of a norm page 68 of SG.** (6 marks)

(ii) **Lemma 4.47 of SG.** 8 marks)

(iii) **Theorem 4.48 of SG.** (3 marks)

(iv) **Theorem 4.50 of SG.** (3 marks)

[20 **marks]**

Question 3

Given the system:

$$\dot{\mathbf{X}}(t) = \begin{pmatrix} t & 2t & 0 \\ \sin t & \frac{1}{t} & 2 \\ 2\sin t & 2\cos t & \frac{1}{t^2} \end{pmatrix} \mathbf{X} + \begin{pmatrix} t \\ \mathrm{e}^t \\ \mathrm{e}^{-t} \end{pmatrix}$$

and the fact that on the interval $[1,4]$ *there are two solutions* $\mathbf{P}$ *and* $\mathbf{Q}$ *with:*

$$\mathbf{P}(2) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \qquad \textit{and } \mathbf{Q}(2) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},$$

find an absolute bound for the error in estimating the solution $\mathbf{P}$ *by the solution* $\mathbf{Q}$*.* ***[6 marks]***

Solution

For the given system we have

$$A(t) = \begin{pmatrix} t & 2t & 0 \\ \sin t & \dfrac{1}{t} & 2 \\ 2\sin t & 2\cos t & \dfrac{1}{t^2} \end{pmatrix} \checkmark$$

So that

$$\begin{aligned} \|A\| &= \max\left\{t + |\sin t| + 2\,|\sin t|\,, 2t + \frac{1}{t} + 2\,|\cos t|\,, 2 + \frac{1}{t^2}\right\} \checkmark \\ &\leq \max\left\{t + 3, 2t + \frac{1}{t} + 2, 2 + \frac{1}{t^2}\right\} \checkmark = \max\{7, 11, 3\} = 11\checkmark, \quad \text{since } 1 \leq t \leq 4 \end{aligned}$$

By Theorem 4.50, we have

$$\begin{aligned} \|\mathbf{P}(t) - \mathbf{Q}(t)\| &\leq \|\mathbf{P}(2) - \mathbf{Q}(2)\| \exp\left(\|A\|\,|t-2|\right) \checkmark \\ &< (2+1+2)e^{11\times 2} = 5e^{22}.\checkmark \end{aligned}$$

[6 **marks]**

Question 4 *Consider the n^{th} order differential equation with constant coefficients*[1]*:*

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y^{(1)} + a_0 y = f(t).$$

Now answer the following questions with respect to this differential equation.

(i) Show that: *(5 marks)*

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{F},$$

where:

$$\mathbf{X} = \begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots - a_{n-1} & \end{pmatrix} \quad \textit{and} \quad \mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}.$$

Note: *The matrix* **A** *is often called a* companion matrix.

[1] Note, according to general convention, $y^{(n)}$ is used to denote the n^{th} order derivative $\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}y$, where y is a function of t. In this notation, $y^{(0)} = y$, $y^{(1)} = \dot{y}$, $y^{(2)} = \ddot{y}$, ...

(ii) Show that if y be a solution of the given differential equation above and $\mathbf{Y} = \begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{pmatrix}$*, then* $\mathbf{Y}$ *is a solution of the system of differential equations in* (i). *(5 marks)*

(iii) Given:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{pmatrix}$$

find a differential equation whose companion matrix is $\mathbf{A}$. *(5 marks)*

(iv) Find the general solution of the differential equation in (iii). *(5 marks)*

[20 marks]

Solution

(i) **Reframing the Equation and definition of companion matrix pages 73 and 74 of SG.** (5 marks)

(ii) **Theorem 5.3 of SG.** (5 marks)

(iii)

$$\begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \end{pmatrix} \checkmark\checkmark\checkmark$$

so $y^{(3)} = -2y^{(1)} + 3y^{(2)} \to y^{(3)} - 3y^{(2)} + 2y^{(1)} = 0$ ✓✓ (5 marks)

(iv) The general

From $y^{(3)} - 3y^{(2)} + 2y^{(1)} = 0$✓the auxiliary equation is $m^3 - 3m^2 + 2m = m(m^2 - 3m + 2) = m(m-1)(m-2) = 0$✓ $\Leftrightarrow m = 0, 1, 2,$✓yielding the complimentary function:

$$y_{C.F.}(t) = c_1 + c_2 e^t + c_3 e^{2t}, \checkmark\checkmark \tag{1}$$

where c_1, c_2 and c_3 are arbitrary constants.✓ (5 marks)

[20 marks]

Question 5

(i) When is a function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ *said to be analytic at a point* t_0*?* *(2 marks)*

(ii) *Find a solution of the system of first order differential equations by using the power series method*

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}, \quad \textit{with} \quad \begin{pmatrix}x\\ y\\ z\end{pmatrix}(0) = \begin{pmatrix}2\\ 0\\ 2\end{pmatrix}$$

(12 marks)

[14 marks]

Solution

(i) **Having continuous derivatives of all order in a neighborhood of t_0 and having a power series expansion about t_0.** (2 marks)

(ii) We have

$$\mathbf{X}(0) = \begin{pmatrix}2\\ 0\\ 2\end{pmatrix} \text{ and } A = \begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}$$

Since all entries are constants, the matrix A is analytic at $t = 0$.✓Thus the sum

$$\sum_{k=0}^{\infty} A_k (t - t_0)^k$$

can yield the constant matrix

$$\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}$$

if and only if $A_1, A_2, A_3, \cdots$ are all the zero matrix.✓Hence, $A_0 = A$ so that

$$\mathbf{X}_n = \frac{1}{n}\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}\mathbf{X}_{n-1} = \frac{1}{n!}\begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}^n \mathbf{X}_0.✓✓$$

Now from

$$A\mathbf{X}_0 = \begin{pmatrix}0\\ 0\\ -2\end{pmatrix} \text{ and } A^2\mathbf{X}_0 = \begin{pmatrix}1 & 1 & -1\\ 0 & 2 & 0\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 0\\ -2\end{pmatrix} = \begin{pmatrix}2\\ 0\\ 2\end{pmatrix}, \quad A^3\mathbf{X}_0 = \begin{pmatrix}0\\ 0\\ -2\end{pmatrix}$$

$$A^4\mathbf{X}_0 = \begin{pmatrix}2\\ 0\\ 2\end{pmatrix}, \quad A^5\mathbf{X}(0) = \begin{pmatrix}0\\ 0\\ -2\end{pmatrix} ✓✓$$

we notice the pattern

$$A^{2k}\mathbf{X}_0 = \begin{pmatrix}2\\ 0\\ 2\end{pmatrix}, \quad k = 0, 1, \cdots ✓ \quad A^{2k+1}\mathbf{X}_0 = \begin{pmatrix}0\\ 0\\ -2\end{pmatrix}, \quad k = 0, 1, \cdots ✓$$

so that

$$\mathbf{X}(t) = \sum_{k=0}^{\infty} \mathbf{X}_k \, t^k \;=\; \sum_{k=0}^{\infty} \mathbf{X}_{2k} \, t^{2k} + \sum_{k=0}^{\infty} \mathbf{X}_{2k+1} \, t^{2k+1} \checkmark$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k} \mathbf{X}(0) \, t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} A^{2k+1} \mathbf{X}(0) \, t^{2k+1} \checkmark$$

$$= \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \checkmark$$

$$\mathbf{X}(t) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cosh(x) + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \sinh(x) \checkmark$$

(12 marks)

[14 marks]

Question 6

(a) State clearly (with all the assumptions and conditions) the classification scheme of critical points of $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$. *(10 marks)*

(b) Classify the critical points of the plane autonomous system corresponding to the second order non-linear differential equation: *(10 marks)*

$$\ddot{x} + \nu(x^2 - 1)\dot{x} + x = 0.$$

[20 marks]

Solution

(i) **The short note on critical points.** (10 marks)

(ii) Let $\dot{x} = y \checkmark$ then $\dot{y} = -x + \nu(1 - x^2)y \checkmark$.
For the critical points : $\dot{x} = 0 = \dot{y} \Rightarrow \ddot{x} = \dot{x} = 0$ so $x = 0$.

Therefore the critical point is $(0,0)$✓. Jacobian is $J = \begin{pmatrix} 0 & 1 \\ -1-2xy\nu & \nu(1-x^2) \end{pmatrix}$✓

$$\begin{aligned} J(0,0) &= \begin{pmatrix} 0 & 1 \\ -1 & \nu \end{pmatrix} ✓ \\ & \tau = \nu \Rightarrow \tau > 0\,(\text{Unstable}) \text{ and } \tau < 0(\text{Stable}) \\ & \Delta = 1 \\ & \tau^2 - 4\Delta = \nu^2 - 4 = (\nu-2)(\nu+2) ✓ \Rightarrow -2 < \nu < 2 \Rightarrow \tau^2 - 4 < 0\,(\text{Spiral}) \\ & \qquad \left.\begin{array}{l} \nu > 2 \\ \nu < -2 \end{array}\right\} \Rightarrow \tau^2 - 4 > 0 \text{node} \\ & \nu > 2: \quad \text{Unstable node}✓ \\ & 0 < \nu < 2: \quad \text{Unstable Spiral}✓ \\ & -2 < \nu < 0: \quad \text{stable Spiral}✓ \\ & \nu < -2: \quad \text{stable node}✓ \\ & \nu = 2: \quad \text{degenerate unstable node.} \\ & \nu = -2: \quad \text{degenerate stable node.} \end{aligned}$$

(10 marks)

[20 **marks**]

Question 2 *Determine whether the system*

$$\dot{x} + x - \dot{y} - y = e^{\frac{t}{2}},$$
$$2\dot{x} - x - 2\dot{y} + y = 5e^{-t}$$

has no solution or infinitely solution. If it has no solution, explain why, else solve $y(t)$ in term of $x(t)$, where $x(t)$ is an arbitrary function of t.

Solution

In operator notation the system becomes

(1) $$(D+1)[x] - (D+1)[y] = e^{\frac{t}{2}}, \checkmark$$
(2) $$(2D-1)[x] - (2D-1)[y] = 5e^{-t} \checkmark$$

Hence $\begin{vmatrix} D+1 & -(D+1) \\ 2D-1 & -(2D-1) \end{vmatrix} \checkmark = -(D+1)(2D-1) + (D+1)(2D-1) = 0 \checkmark$ so that system is degenerate$\checkmark\checkmark$. Furthermore, since

$$\begin{vmatrix} D+1 & e^{\frac{t}{2}} \\ 2D-1 & 5e^{-t} \end{vmatrix} \checkmark = \begin{vmatrix} e^{\frac{t}{2}} & -(D+1) \\ 5e^{-t} & -(2D-1) \end{vmatrix} \checkmark = (D+1)[5e^{-t}] - (2D-1)e^{\frac{t}{2}} \checkmark = 5e^{-t} - 5e^{-t} - e^{\frac{t}{2}} + e^{\frac{t}{2}} = 0. \checkmark\checkmark\checkmark$$

The system has infinity many solutions.$\checkmark\checkmark$
Let $x(t) = g(t)\checkmark$ where $g(t)$ is an arbitrary function$\checkmark$. Then $2\times$(1)$-$(2) $: 3x(t) - 3y(t) = 2e^{\frac{t}{2}} - 5e^{-t}\checkmark\checkmark$
So that

$$y(t) = g(t) - \frac{2}{3}e^{\frac{t}{2}} + \frac{5}{3}e^{-t}\checkmark\checkmark$$

[20 **marks]**