MAT2611 LINEAR ALGEBRA May/June 2016

SOLUTIONS

Please note: any **fundamental error** is grounds for **no marks** being awarded for an answer.

For some questions, different methods may be used to obtain a correct answer (unless the question specifies the method to be used). Some questions do not have a unique solution. In both cases, full marks will be awarded for answers which answer the given question and are mathematically correct.

QUESTION 1

This question is a **multiple choice** question and should be answered in the **answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book. Write only the *number* for your answer.

1.1) 3
1.2) 1
1.3) 2
1.4) 1
1.5) 4
1.6) 2
1.7) 3
1.8) 4

2 marks each

(1.1) Consider the set (2)

$$X := \{\spadesuit\}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which of the following statements are true in X ?

A. for all $\mathbf{x} \in X$: $-\mathbf{x} = \spadesuit$

B. for all $\mathbf{x} \in X$: $-\mathbf{x} = \mathbf{x}$

C. $\mathbf{0} = 0$

D. $\mathbf{0} = (0, 0)$

Choose from the following:

1. A
2. B
3. A and B
4. C or D
5. None of the above.

Answer: 3

(1.2) Which of the following are subspaces of M_{22} with the usual operations ? (2)

A. $\text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} : a \geq 0 \right\}$

C. $\left\{ \begin{bmatrix} a & -1 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B and C.
4. All of A, B and C.
5. None of the above.

Answer: 1

(1.3) Which of the following sets are linearly independent? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in M_{22}

B. $\{ (1,0,1), (0,1,0), (1,1,-1) \}$ in $\mathbb{R}^3$

C. $\{ 1-x, 1-x^2, 1-x+x^2 \}$ in P_2

Select from the following:

1. Only A and C.
2. Only B and C.
3. Only B.
4. Only C.
5. None of the above.

Answer: 2

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Both A and B.
2. Only A.
3. Only B.
4. None of the above.

Answer: 1

(1.5) Which of the following statements are true: (2)

A. $\dim\left(\operatorname{span}\left\{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\right\}\right) = 2$ in M_{22}

B. $\dim(\operatorname{span}\{(1,0,1), (0,1,0), (1,1,-1)\}) = 3$ in $\mathbb{R}^3$

C. $\dim(\operatorname{span}\{1-x, 1-x^2, 1-x+x^2\}) = 2$ in P_2

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only A and B.
5. None of the above.

Answer: 4

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix}1 & 1 & -1\\0 & 1 & -1\\1 & 0 & 0\\1 & -2 & 2\end{bmatrix}$? (2)

A. $\{\begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\end{bmatrix}\}$

B. $\{\begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & -1\end{bmatrix}\}$

C. $\{\begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & -2 & 2\end{bmatrix}\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Only C.
5. None of the above.

Answer: 2

(1.7) Which of the following sets are a basis for the null space of $\begin{bmatrix}1 & 1 & -1\\0 & 1 & -1\\1 & 0 & 0\\1 & -2 & 2\end{bmatrix}$? (2)

A. $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

B. $\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

C. $\left\{\begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$

Select from the following:

1. Only A.

2. Only B.
3. Both B and C.
4. All of A, B and C.
5. None of the above.

Answer: 3

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.
2. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
3. $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.
4. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.
5. None of the above.

Answer: 4

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QUESTION 2

Consider the vector space M_{22}.

(2.1) Show that (12)

$$\langle A, B\rangle = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right)$$

is an inner product on M_{22}.

- *For all* $A, B \in M_{22}$

$$\langle A, B\rangle = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right) = \text{tr}\left(BA^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} BA^T\right) = \langle B, A\rangle \checkmark^{2}$$

  where we used Theorem TI and Theorem CT.

- *For all* $k \in \mathbb{R}$ *and* $A \in M_{22}$ *we have*

$$\langle kA, B\rangle = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} (kA)B^T\right) = k\,\text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right) = k\langle A, B\rangle \checkmark^{2}$$

  since $\text{tr}(kA) = k\,\text{tr}(A)$.

- *For all* $A, B, C \in M_{22}$

$$\begin{aligned}\langle A, B+C\rangle &= \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A(B+C)^T\right) = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A(B^T+C^T)\right) \\ &= \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AC^T\right) \\ &= \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right) + \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AC^T\right) = \langle A, B\rangle + \langle A, C\rangle \checkmark^{4}\end{aligned}$$

  since $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$.

- *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then*

$$\langle A, A\rangle = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AA^T\right) = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}\right) = 2a^2+2b^2+c^2+d^2 \geq 0 \checkmark^2$$

and $\langle A, A\rangle = 0$ *if and only if* $a = b = c = d = 0$ *(since* $a^2, b^2, c^2, d^2 \geq 0$*).* ✓2

Alternative:

Note that $\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\right) = 2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4.$

- *For all* $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in M_{22}$

$$\begin{aligned}\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle &= 2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4 \\ &= 2b_1a_1 + 2b_2a_2 + b_3a_3 + b_4a_4 = \left\langle \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right\rangle.\end{aligned}$$

- *For all* $k \in \mathbb{R}$ *and* $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_{22}$

$$\begin{aligned}\left\langle k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle \\ &= 2(ka_1)b_1 + 2(ka_2)b_2 + (ka_3)b_3 + (ka_4)b_4 \\ &= k(2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4) = k\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle.\end{aligned}$$

- *For all* $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \in M_{22}$

$$\begin{aligned}\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{bmatrix} \right\rangle \\ &= 2a_1(b_1+c_1) + 2a_2(b_2+c_2) + a_3(b_3+c_3) + a_4(b_4+c_4) \\ &= (2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4) + (2a_1c_1 + 2a_2c_2 + a_3c_3 + a_4c_4) \\ &= \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right\rangle.\end{aligned}$$

- *Let* $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. *Then*

$$\langle A, A\rangle = 2a_1^2 + 2a_2^2 + a_3^2 + a_4^2 \geq 0$$

and $\langle A, A\rangle = 0$ *if and only if* $a_1 = a_2 = a_3 = a_4 = 0$ *(since* $a_1^2, a_2^2, a_3^2, a_4^2 \geq 0$*).*

(2.2) Prove that if $A, B \in M_{22}$, where $A, B \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, are orthogonal to each other with respect to the inner product **defined in 2.1** above, then $\{A, B\}$ is a linearly independent set. (6)

Suppose $c_1A + c_2B = 0$ *where* $c_1, c_2 \in \mathbb{R}$. *Since* A *and* B *are orthogonal to each other we have* $\langle A, B\rangle = \langle B, A\rangle = 0$.✓

$$\begin{aligned}c_1A + c_2B = 0 &\Rightarrow \langle A, c_1A + c_2B\rangle = \langle A, 0\rangle \checkmark \\ &\Rightarrow c_1\langle A, A\rangle + c_2\langle A, B\rangle = 0 \checkmark \\ &\Rightarrow c_1\langle A, A\rangle = 0 \checkmark\end{aligned}$$

[TURN OVER]

$$\Rightarrow c_1 = 0 \checkmark$$

since $\langle A, A\rangle \neq 0$. *Similarly*

$$c_1A + c_2B = 0 \Rightarrow \langle B, c_1A + c_2B\rangle = \langle B, 0\rangle$$
$$\Rightarrow c_2 = 0. \checkmark$$

Thus $\{A, B\}$ *is a linearly independent set.*

(2.3) Apply the Gram-Schmidt process to the following subset of M_{22}: (12)

$$\left\{ \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.
Let

$$u_1 := \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}, \quad u_2 := \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \quad u_3 := \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$v_1 := u_1 = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} \checkmark$$
$$\langle v_1, v_1\rangle = \operatorname{tr}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}^T\right) = 8 \checkmark$$
$$\langle u_2, v_1\rangle = \operatorname{tr}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}^T\right) = 4 \checkmark$$
$$v_2 := u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1(x) \checkmark$$
$$= \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} - \frac{4}{8}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix} \checkmark$$
$$\langle v_2, v_2\rangle = \frac{1}{4}\operatorname{tr}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}^T\right) = 2 \checkmark$$
$$\langle u_3, v_1\rangle = \operatorname{tr}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}^T\right) = 4 \checkmark$$
$$\langle u_3, v_2\rangle = \frac{1}{2}\operatorname{tr}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}^T\right) = 0 \checkmark$$
$$v_3 := u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \checkmark$$
$$= \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} - \frac{4}{8}\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} - \frac{0}{2}\cdot\frac{1}{2}\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix}. \checkmark$$

Thus we have the orthogonal basis

$$\left\{ \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}, \begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} \right\}. \checkmark 2$$

(2.4) Let V be a vector space with zero vector $\mathbf{0}$ and let $\langle\cdot,\cdot\rangle$ denote an inner product on V. Prove that $\langle \mathbf{0}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in V$. (4)

Since $\mathbf{0} = 0 \cdot \mathbf{0}$ ✓2 *(Theorem VZ) we have* $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0 \cdot \mathbf{0}, \mathbf{v} \rangle = 0 \langle \mathbf{0}, \mathbf{v} \rangle = 0$ ✓2 *by IP2.*

Alternative:

Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ *(VS4) we have* $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0} + \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle + \langle \mathbf{0}, \mathbf{v} \rangle$ *by IP3 and IP1, so that* $\langle \mathbf{0}, \mathbf{v} \rangle = 0$.

Alternative:

Since the Cauchy-Schwarz inequality yields $|\langle \mathbf{0}, \mathbf{v} \rangle| \leq \sqrt{\langle \mathbf{0}, \mathbf{0} \rangle \langle \mathbf{v}, \mathbf{v} \rangle} = 0$ *by IP4b, it follows that* $|\langle \mathbf{0}, \mathbf{v} \rangle| = 0$. *Thus* $\langle \mathbf{0}, \mathbf{v} \rangle = 0$.

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QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

Row reduction of A yields

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_1)$$

which is in upper triangular form, with two nonzero rows. Hence the rank is 2, and the nullity is $3 - 2 = 1$. ✓2

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda(\lambda - 1)^2 = 0.$$

The characteristic equation is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\right) ✓ = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & 0 \\ -1 & -1 & \lambda \end{vmatrix}$$
$$= (\lambda - 1)^2 \lambda = 0 ✓2.$$

(3.3) Find bases for the eigenspaces of A. (14)

From the characteristic equation we obtain the eigenvalues 0 (twice), and 1 ✓2*. For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ✓2$$

for $x, y, z \in \mathbb{R}$. *Clearly* $x = -y = 0$. *We find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}. ✓2$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. ✓2$$

For the eigenspace corresponding to the eigenvalue 1 we solve

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark 2$$

for $x, y, z \in \mathbb{R}$. Obviously $y = 0$ and $x = z$. The corresponding eigenspace is

$$\left\{ \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} : x \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : x \in \mathbb{R} \right\}. \checkmark 2$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}. \checkmark 2$$

(3.4) For each eigenvalue, determine the algebraic and geometric multiplicity. Is A diagonalizable? (5)

The algebraic multiplicity of $\lambda = 0$ is 2,✓ and the geometric multiplicity is 1.✓ Thus A is not diagonalizable (Theorem DM).✓ The algebraic multiplicity of $\lambda = 1$ is 1,✓ and the geometric multiplicity is 1.✓

(3.5) Prove or disprove: (2)

If B is a 2×2 matrix, then B is diagonalizable if and only if B^2 is diagonalizable.

The statement is false, for example the matrix $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable, but $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable (and diagonal).✓2

(3.6) Let B be an $n \times n$ matrix. Prove that $B + B^T$ is diagonalizable. (2)

Since $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$, $B + B^T$ is symmetric✓ and consequently diagonalizable (Theorem DS).✓

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QUESTION 4

Let $T : \mathbb{R}^3 \to M_{22}$ be defined by $T(x, y, z) = \begin{bmatrix} x & y \\ z & x \end{bmatrix}$.

(4.1) Show that T is a linear transformation. (4)

Let $k \in \mathbb{R}$ and $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$.

- $T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ z_1 + z_2 & x_1 + x_2 \end{bmatrix}$
  $= \begin{bmatrix} x_1 & y_1 \\ z_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ z_2 & x_2 \end{bmatrix} = T(x_1, y_1, z_1) + T(x_2, y_2, z_2).$ ✓2
- $T(k \cdot (x_1, y_1, z_1)) = T(kx_1, ky_1, kz_1) = \begin{bmatrix} kx_1 & ky_1 \\ kz_1 & kx_1 \end{bmatrix} = k \begin{bmatrix} x_1 & y_1 \\ z_1 & x_1 \end{bmatrix} = kT(x_1, y_1, z_1).$ ✓2

(4.2) Find the matrix representation $[T]_{B_2, B_1}$ of T relative to the basis (8)

$$B_1 = \{ (1, 0, 1), (0, 1, 0), (1, 0, -1) \}$$

in $\mathbb{R}^3$ and the basis

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, ordered from left to right.

From

$$T(1,0,1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark 2$$

$$T(0,1,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark 2$$

$$T(1,0,-1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark 2$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}. \checkmark 2$$

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

The range of T is

$$\begin{aligned} R(T) &= \{ T(x,y,z) \, : \, x,y,z \in \mathbb{R} \} \\ &= \left\{ \begin{bmatrix} x & y \\ z & x \end{bmatrix} \, : \, x,y,z \in \mathbb{R} \right\}. \checkmark 2 \end{aligned}$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_{22}$ *but* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin R(T)$, *$T$ is not onto.* $\checkmark 2$

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

$$\begin{aligned} \ker(T) &= \left\{ (x,y,z) \in \mathbb{R}^3 \, : \, T(x,y,z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \left\{ (x,y,z) \in \mathbb{R}^3 \, : \, \begin{bmatrix} x & y \\ z & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \{ (0,0,0) \}. \checkmark 2 \end{aligned}$$

We have a zero-dimensional space and the nullity of T is 0. $\checkmark 2$

(4.5) Is T one-to-one? Motivate your answer. (2)

Yes, since $\text{nullity}(T) = 0$ *(or equivalently* $\ker(T) = \{ (0,0,0) \}$, *Theorem TO).* $\checkmark 2$

[22]

TOTAL MARKS: [100]