Tutorial letter 103/2/2018

Distribution Theory I

STA1503

Semester 2

Department of Statistics

TRIAL EXAMINATION PAPER

1. Trial Examination Papar
2. Trial Examination Paper Solutions

1 Trial Examination Paper Part I

QUESTION 1 **[Total 15 marks]**

If A and B are *independent* events with $P(A) = 0.50$, and $P(B) = 0.20$, find

(a) $P(A \cap B)$ (3)

(b) $P(\overline{A} \cup \overline{B})$ (3)

(c) $P(\overline{A} \cap \overline{B})$ (4)

(d) $P(\overline{A}/B)$ (5)

QUESTION 2 **[Total 15 marks]**

Let Y be a binomial random variable with $n = 10$ and $p = 0.2$.

(a) Use Table 1, Appendix 3, to obtain $P(2 < Y < 5)$ and $P(2 \leq Y < 5)$. Are the probabilities that Y falls in the interval $(2, 5)$ and $[2, 5)$ equal? Why or why not? (5)

(b) Use Table 1, Appendix 3, to obtain $P(2 < Y < 5)$ and $P(2 \leq Y \leq 5)$. Are these two probabilities equal? Why or why not? (5)

(c) If Y is continuous and $a < b$, then $P(a < Y < b) = P(a \leq Y < b)$. Does the result in part (a) contradict this claim? Why? (5)

QUESTION 3 **[Total 10 marks]**

These are all questions on chapter 3: **Discrete random variables and their probability distributions.**
To verify the accuracy of their accounting entries, a company uses auditors for verification on a regular basis. The company's employees make erroneous entries 5% of the time. Suppose that an auditor randomly checks three entries.

(a) Find the probability distribution for Y, the number of errors detected by the auditor. (5)

(b) Find the probability that the auditor will detect more than one error. (5)

QUESTION 4 **[Total 20 marks]**

Let Y be a random variable with $p(y)$ given in the accompanying table.

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

Calculate the following:

(a) $E(Y)$ (5)

(b) $E\left(\frac{1}{Y}\right)$ (5)

(c) $E(Y^2 - 1)$ (5)

(d) $V(Y)$ (5)

QUESTION 5 **[Total 20 marks]**

Let X be random variable with probability density function given by

$$f_X(x) = \begin{cases} c, & 0 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find c (5)

(b) Find $E(X)$ and $Var(X)$ (8)

(c) Find $E(4X + 5)$ and $Var(6X + 2)$ (7)

QUESTION 6 **[Total 20 marks]**

Let Y be a random variable with density function

$$f(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the cumulative distribution function of Y (6)

(b) Find the density function of $4Y - 1$ (8)

(c) Find $E(Y)$ (6)

QUESTION 7 **[Total 50 marks]**

Let Y_1 and Y_2 be continuous random variables with joint density function

$$f(y_1, y_2) = \begin{cases} cy_1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find

(a) The value of c that makes $f(y_1, y_2)$ a joint probability density function. (6)

(b) The joint cumulative distribution function $F(y_1, y_2)$. (6)

(c) The marginal density function of Y_1, $f_1(y_1)$. (6)

(d) The marginal density function of Y_2, $f_2(y_2)$. (6)

(e) $E(Y_1Y_2)$ (6)

(f) $E(Y_1)$ (6)

(g) $Var(Y_2)$ (8)

(h) $Cov(Y_1, Y_2)$ and draw a conclusion. (6)

Total marks [150]

2 Trial Examination Paper Part II

QUESTION 1 **[Total 40 marks]**

(a) A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year. (10)

(b) The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%.

Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work.

Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist. (10)

(c) You are given $P[A \cup B] = 0.7$ and $P[A \cup B\prime] = 0.9$. Determine $P[A]$. (10)

(d) An urn contains 10 balls : 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44.

Calculate the number of blue balls in the second urn. (10)

QUESTION 2 **[Total 55 marks]**

(a) An auto insurance company has$10,000$ policyholders. Each policyholder is classified as

(i) young or old;
(ii) male or female; and
(iii) married or single.

Of these policyholders,$3000$are young,4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single? (10)

(b) A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease.

Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease. (10)

(c) An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year.

Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy.

Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year. (10)

(d) Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist. (10)

(e) An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.

(ii) 70% of the customers insure more than one car.

(iii) 20% of the customers insure a sports car.

(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car. (5)

(f) An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.

(ii) 64% of the customers insure more than one car.

(iii) $20\%$of the customers insure a sports car.

(iv) Of those customers who insure more than one car, 15% insure a sports car.

What is the probability that a randomly selected customer insures exactly one car, and that car is not a sports car? (10)

QUESTION 3 **[Total 25 marks]**

(a) A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is $\frac{3}{5}$. The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months.

Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs. (5)

(b) A group insurance policy covers the medical claims of the employees of a small company. The value, V, of the claims made in one year is described by $V = 100,000Y$ where Y is a random variable with density function

$$f_Y(y) = \begin{cases} k(1-y)^4 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

where k is a constant. What is the conditional probability that V exceeds $40,000$, given that V exceeds $10,000$? (10)

(c) A random variable X has the cumulative distribution function

$$\begin{cases} 0 & for \quad 1 < x \\ \dfrac{x^2 - 2x - 2}{2} & for \quad 1 \leq x < 2 \\ 1 & for \quad x \geq 2 \end{cases}$$

Calculate the variance of X. (5)

(d) Let X represent the age of an insured automobile involved in an accident. Let Y represent the length of time the owner has insured the automobile at the time of the accident. X and Y have joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{64}(10 - xy^2) & 2 \leq x \leq 10 \text{ and , } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the expected age of an insured automobile involved in an accident. (5)

Total marks [120]

3 Trial Examination Solutions Part I

QUESTION 1

A and B are independent, $P(A) = 0.5$, $P(B) = 0.2$:

(a) $P(A \cap B) = P(A) \cdot P(B)$ (because of independency) $= 0.5.0.2 = 0.1$ (3)

(b) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.1 = 0.9$ (3)

(c)

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= P(\overline{A \cup B}) = 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - (0.5 + 0.2 - 0.1) = 0.4 \end{aligned}$$

(4)

(d)

$$\begin{aligned} P(\bar{A}|B) &= \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(\bar{A}) \cdot P(B)}{P(B)} \\ &= \frac{(1 - P(A)) \cdot P(B)}{P(B)} \\ &= \frac{0.5 \cdot 0.2}{0.2} = 0.5 \end{aligned}$$

or directly,

$$P(\bar{A}|B) = P(\bar{A}) = 1 - P(A) = 0.5$$

In both solutions we use the fact that if A and B are independent, then of course A^c and B must also be independent. (5)

[Total marks: 15]

QUESTION 2

$$Y \sim Bin(n = 10,\ \ p = 0.2) :$$

The table to use is Table 1, Appendix 3 of the textbook; and for $n = 10$ we choose table (b) on page 839. For $p = 0.2$, we look at the fourth column of values.
Note that the table given probabilities of the type:

$$P(Y \leq a)$$

Therefore we must express the gives probabilities we wish to evaluate in terms of these types of probabilities, keeping in mind that:
$P(Y < a)$ and $P(Y \leq a)$ are not necessarily the same thing. Note that of cause $P(a < Y \leq b) = P(Y \leq b) - P(Y \leq b)$, this therefore will also be easy to evaluate.

(a)

$$\begin{aligned}P(2 < Y < 5) &= P(2 < Y \leq 4)\\ &= P(Y \leq 4) - P(Y \leq 2)\\ &= 0.967 - 0.678 = 0.289 \qquad \text{(from table)}\\ P(2 \leq Y < 5) &= P(1 < Y \leq 4)\\ &= P(Y \leq 4) - P(Y \leq 1)\\ &= 0.967 - 0.376\\ &= 0.591\end{aligned}$$

The values are not equal because $Y = 2$ is included in the second interval but in the first one; the difference between the probabilities is:

$$0.591 - 0.289 = 0.302$$

Which is $P(Y = 2)$ (5)

(b)

$$\begin{aligned}P(2 < Y < 5) &= P(2 < Y \leq 4) = 0.289\\ P(2 \leq Y \leq 5) &= P(1 < Y \leq 5) = P(Y \leq 5) - P(Y \leq 1)\\ &= 0.994 - 0.376 = 0.618\end{aligned}$$

The values are not the same and the reason is because the second probability includes the values $Y = 2$ and $Y = 5$, while the first probability does not. The difference between the probabilities is equal to $P(Y = 2) + P(Y = 5)$. (5)

(c) No, there is no contradiction; the Binomial distribution is not a continuous distribution but rather discrete and therefore $P(a < Y < b) = P(a \leq Y < b)$ also not have to hold.

$P(a < Y < b) = P(a \leq Y < b)$ holds for continuous distribution and does not hold for discrete distribution. (5)

[Total marks: 15]

QUESTION 3

(a) Assuming the three chosen entries are independent of each other, each with an error with the probability 0.05, then Y, the number of errors detected by the auditor will have Binomial distribution with parameters $n = 3$ and $p = 0.05$ (5)

(b)

$$\begin{aligned}P(Y > 1) &= 1 - P(Y \leq 0) - P(Y = 1)\\ &= 1 - \binom{3}{0} p^0(1-p)^3 - \binom{3}{1} p^1(1-p)^2\\ &= 1 - 1(0.05)^0(0.95)^3 - 3(0.05)(0.95)^2\\ &= 0.00725\end{aligned}$$

or alternatively,

$$\begin{aligned} P(Y>1) &= P(Y=2)+P(Y=3) \\ &= \binom{3}{2} p^2(1-p)^1 + \binom{3}{3} p^3(1-p)^0 \\ &= 3(0.05)^2(0.95)^1 + 1(0.05)^3(0.95)^0 \\ &= 0.0725 \end{aligned}$$

(5)

[Total marks: 10]

QUESTION 4

The probability function $P(y) = P(Y = y)$ is as follows:

y	1	2	3	4
$p(y)$	0.4	0.3	0.2	0.1

(a)

$$\begin{aligned} E(Y) &= \sum_{y=1}^{4} yP(y) = 1\cdot 0.4 + 2\cdot 0.3 + 3\cdot 0.2 + 4\cdot 0.1 \\ &= 2.0 \end{aligned}$$

(5)

(b)

$$\begin{aligned} E\left(\frac{1}{Y}\right) &= \sum_{y=1}^{4} \frac{1}{y}P(y) = \frac{1}{1}\cdot 0.4 + \frac{1}{2}\cdot 0.3 + \frac{1}{3}\cdot 0.2 + \frac{1}{4}\cdot 0.1 \\ &= \frac{77}{120} \end{aligned}$$

(5)

(c)

$$\begin{aligned} E(Y^2-1) &= E(Y^2) - 1 = \left(\sum_{y=1}^{4} y^2 p(y)\right) - 1 \\ &= (1^2\cdot 0.4 + 2^2\cdot 0.3 + 3^2\cdot 0.2 + 4^2\cdot 0.1) - 1 \\ &= 5 - 1 = 4 \end{aligned}$$

(5)

(d)

$$V(Y) = E(Y^2) - (E(Y))^2$$

where

$$E(Y) = 2, \qquad E(Y^2) = 5$$

As calculated in (a) and (c). Therefore:

$$V(Y) = 5 - (2)^2 = 1$$

Alternatively,

$$\begin{aligned} V(Y) &= E(Y - E(Y)^2) \\ &= E((Y-2)^2) \\ &= \sum_{i=1}^{4}(y-2)^2 P(y) \\ &= (1-2)^2 \cdot 0.4 + (2-2)^2 \cdot 0.3 + (3-2)^2 \cdot 0.2 + (4-2)^2 \cdot 0.1 \\ &= 1 \end{aligned}$$

(5)

[Total marks: 20]

QUESTION 5

$$f_X(x) = \begin{cases} C, & 0 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) The value of c can be determined from the fact that we know that for f_X to be a density function, it must integrate to the value 1 when integrated over all possible x-values. Here, we therefore get:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_0^2 c dx \Leftrightarrow 1 = cx|_0^2 \\ 1 &= 2c \\ \therefore \quad c &= \frac{1}{2} \end{aligned}$$

(5)

(b)

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^2 x \cdot \frac{1}{2}dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^2 \\ &= \left.\left(\frac{1}{4}x^2\right)\right|_0^2 = \frac{1}{4}2^2 - \frac{1}{4}0^2 = 1 \end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^2 x^2 \cdot \frac{1}{2}dx = \frac{1}{2}\left.\frac{x^3}{3}\right|_0^2 \\
&= \left.\left(\frac{1}{6}x^3\right)\right|_0^2 = \frac{1}{6}2^3 - \frac{1}{6}0^3 = \frac{8}{6} = \frac{4}{3} \\
Var(X) &= E(X^2) - (E(X))^2 = \frac{4}{3} - 1^2 = \frac{1}{3}
\end{aligned}$$

Alternative, we can recognize the distribution as being the uniform distribution on the interval $(0, 2)$ for which $E(X) = \frac{0+2}{2} = 1$ and $Var(X) = \frac{(2-0)^2}{12} = \frac{4}{12} = \frac{1}{3}$. (8)

(c)

$$\begin{aligned}
E(4X+5) &= 4E(X) + 5 = 4 \cdot 1 + 5 = 9 \\
Var(6X+2) &= 6^2 \cdot Var(X) = 36 \cdot \frac{1}{3} = 12
\end{aligned}$$

(7)

[Total mark 20]

QUESTION 6

$$f_Y(y) = \begin{cases} 2y, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(a) The cumulative distribution function is found by integrating the density function.

$$F_Y(y) = \int_{-\infty}^{y} f_Y(a)da$$

Since the density function is defined piecewise, we will also find the distribution function piecewise.

If $y < 0$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(a)da = \int_{-\infty}^{y} 0da = 0$$

If $0 \le y \le 1$,

$$\begin{aligned}
F_Y(y) &= \int_{-\infty}^{y} f_Y(a)da = \int_{-\infty}^{0} 0da + \int_0^y 2ada \\
&= \left.(a^2)\right|_0^y = y^2
\end{aligned}$$

$$F_y(y) = \begin{cases} \int_{-\infty}^{y} 0dt = 0, & -\infty < y < 0 \\ 0 + \int_0^y 2ada = y^2, & 0 \le y \le 1 \\ 0 + \int_0^1 2ada + \int_1^y 0da = 1, & y > 1 \end{cases}$$

If $y > 1$

$$\begin{aligned} F_X(y) &= \int_{-\infty}^{y} f_Y(a)da = \int_{-\infty}^{0} 0da + \int_{0}^{1} 2ada + \int_{1}^{y} 0da \\ &= (a^2)|_0^1 = 1 \end{aligned}$$

That is:

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ y^2, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

(6)

(b) Let $U = 4Y - 1$. The mapping $u = h(y) = uy - 1$ is increasing, so we can use the transformation method. The inverse mapping is:

$$y = \frac{1}{4}(u+1) = h^{-1}(u)$$

with derivative:

$$\frac{d}{du}(h^{-1}(u)) = \frac{1}{4}$$

Therefore the density function of U is:

$$\begin{aligned} f_U(u) &= f_Y(h^{-1}(u))|\frac{d}{du}h^{-1}(u)| \\ &= \begin{cases} 2\left(\frac{1}{4}\right)(u+1))\frac{1}{4}, & 0 \le \frac{1}{4}(u+1) \le 1 \\ 0. & \text{elsewhere} \end{cases} \\ &= \begin{cases} \frac{1}{8}(u+1), & 1 \le u \le 3 \\ 0. & \text{elsewhere} \end{cases} \end{aligned}$$

Alternatively we can use the distribution function method:

$$F_U(u) = P(U \le u) = P(4Y - 1 \le u) = P\left(Y \le \frac{u+1}{4}\right) = F_Y\left(\frac{u+1}{4}\right)$$

Since we already have the distribution function F_Y, there is no need to integrate f_Y again. We get:

$$\begin{aligned} F_U(u) = F_Y\left(\frac{u+1}{4}\right) &= \begin{cases} 0, & \frac{u+1}{4} < 0 \\ \left(\frac{u+1}{4}\right)^2, & 0 \le \frac{u+1}{4} \le 1 \\ 1, & \frac{u+1}{4} > 1. \end{cases} \\ &= \begin{cases} 0, & u < -1 \\ \left(\frac{u+1}{4}\right)^2, & -1 \le u \le 3 \\ 1, & u > 3. \end{cases} \end{aligned}$$

Next, we differentiate this with respect to u to find the corresponding density function.

$$f_u(u) = \frac{d}{du}F_y(u) = \begin{cases} 0, & u < -1 \\ \frac{1}{4}\left(\frac{u+1}{4}\right)^2, & -1 \le u \le 3 \\ 0, & u > 3 \end{cases}$$

$$= \begin{cases} \frac{1}{8}(u+1), & -1 \le u \le 3 \\ 0, & \text{elsewhere.} \end{cases}$$

(8)

(c)

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{e}^{1} y \cdot 2y dy \\ &= \int_{0}^{1} 2y^2 dy \\ &= \frac{2}{3}(y^3)|_0^1 = \frac{2}{3} \end{aligned}$$

(6)

[Total marks: 20]

QUESTION 7

(a) $f(y_1, y_2)$ is a joint probability density function if

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 dy_2 &= 1 \\ \text{Then, } \int_0^1\int_0^1 c y_1 dy_1 dy_2 &= 1 \\ c\int_0^1 \left.\frac{y_1^2}{2}\right|_0^1 dy_2 &= 1 \\ \frac{c}{2}\int_0^1 dy_2 &= 1 \\ \frac{c}{2} y_2\Big|_0^1 &= 1 \\ \frac{c}{2} &= 1 \\ c &= 2 \end{aligned}$$

Therefore,

$$f(y_1, y_2) = \begin{cases} 2y_1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(6)

(b)

$$\begin{aligned} F(y_1, y_2) &= \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f(y_1, y_2)\, dy_1 dy_2 \\ &= \int_0^{y_1} \int_0^{y_2} 2t_1 dt_2 dt_1 \\ &= 2 \int_0^{y_1} t_1 t_2 \Big|_0^{y_2} dt_1 \\ &= 2 \int_0^{y_1} y_2 t_2 dt_1 \\ &= 2y_2 \left. \frac{t_1^2}{2} \right|_0^{y_1} \\ &= y_2 y_1^2 \\ &= y_1^2 y_2. \end{aligned}$$

$$f(y_1, y_2) = \begin{cases} 0, & y_1 \le 1,\ y_2 \le 1 \\ y_1^2 y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 1, & y_1 > 0,\ y_2 > 1 \end{cases}$$

(6)

(c)

$$\begin{aligned} f_1(y_1) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 \\ &= \int_0^1 2y_1 dy_2 \\ &= 2y_1 y_2|_0^1 \\ &= 2y_1 \end{aligned}$$

$$f_1(y_1) = \begin{cases} 2y_1, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(6)

(d)

$$\begin{aligned} f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 \\ &= \int_0^1 2y_1 dy_1 \\ &= 2 \left. \frac{y_1^2}{2} \right|_0^1 \end{aligned}$$

$$f_2(y_2) = \begin{cases} 1, & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(6)

(e)

$$\begin{aligned}
E(Y_1Y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1y_2 f(y_1, y_2)\, dy_1dy_2 \\
&= \int_0^1\int_0^1 y_1y_2 2y_1 dy_1dy_2 \\
&= \int_0^1\int_0^1 2y_1^2y_2dy_1dy_2 \\
&= 2\int_0^1 y_2 \left.\frac{y_1^3}{3}\right|_0^1 dy_1dy_2 \\
&= \frac{2}{3}\int_0^1 y_2dy_2 \\
&= \frac{2}{3}\left.\frac{y_2^2}{2}\right|_0^1 \\
&= \frac{2}{3}\cdot\frac{1}{2} \\
&= \frac{1}{3}
\end{aligned}$$

(6)

(f)

$$\begin{aligned}
E(Y_1) &= \int_{-\infty}^{\infty} y_1 f_1(y_1)\, dy_1 \\
&= \int_0^1 y_1 2y_1 dy_1 \\
&= 2\int_0^1 y_1^2 dy_1 \\
&= 2\cdot\left.\frac{y_1^3}{3}\right|_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

(6)

(g)

$$\begin{aligned}
Var\left(Y_2\right) &= E\left(Y_2^2\right) - \left[E\left(Y_2\right)\right]^2 \\
E\left(Y_2\right) &= \int_{-\infty}^{\infty} y_2 f_2\left(y_2\right) dy_2 = \int_0^1 y_2 1 dy_2 = \left.\frac{y_2^2}{2}\right|_0^1 = \frac{1}{2} \\
E\left(Y_2^2\right) &= \int_{-\infty}^{\infty} y_2^2 f_2\left(y_2\right) dy_2 = \int_0^1 y_2^2 \cdot 1 dy_2 = \left.\frac{y_2^3}{3}\right|_0^1 = \frac{1}{3} \\
Var\left(Y_2\right) &= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \\
&= \frac{1}{3} - \frac{1}{4} \\
&= \frac{4-3}{12} \\
&= \frac{1}{12} \\
Var\left(Y_2\right) &= \frac{1}{12}
\end{aligned}$$

(8)

(h)

$$\begin{aligned}
Cov\left(Y_1, Y_2\right) &= E\left(Y_1, Y_2\right) - E\left(Y_1\right) \cdot E\left(Y_2\right) \\
&= \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{2} \\
&= \frac{1}{3} - \frac{1}{3} \\
&= 0
\end{aligned}$$

Conclusion: Y_1 and Y_2 are independent. (6)

[Total marks: 50]

4 Trial Examination Solutions Part II

QUESTION 1

(a) Let

$G =$ event that a viewer watched gymnastics.

$B =$ event that a viewer watched baseball.

$S =$ event that a viewer watched soccer.

Then we want to find

$$\begin{aligned}
& P\left[(G \cup B \cup S)^C\right] \\
= \;& 1 - P\left(G \cup B \cup S\right) \\
= \;& 1 - \left[P\left[(G) + P\left(B\right) + P\left(S\right) - P\left(G \cap B\right) - P\left(G \cap S\right) - P\left(B \cap S\right) + P\left(G \cap B \cap S\right)\right]\right] \\
= \;& 1 - (0.28 + 0.29 + 0.19 + 0.14 - 0.10 - 0.12 + 0.08) \\
= \;& 1 - 0.48 \\
= \;& 0.52
\end{aligned}$$

(10)

(b) Let

$R =$ event of referral to a specialist.

$L =$ event of Lab work.

we want to find

$$\begin{aligned}
P[R \cap L] &= P[R] + P[L] - P[R \cup L] \\
&= P[R] + P[L] - 1 + P[\sim (R \cup L)] \\
&= P[R] + P[L] - 1 + P[\sim R \cap \sim L] \\
&= 0.30 + 0.40 - 1 + 0.35 \\
&= 0.05
\end{aligned}$$

(10)

(c) First note

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

$P\left[A \cup B'\right] = P[A] + P\left[B'\right] - P\left[A \cap B'\right]$

Then add these two equations to get

$$\begin{aligned}
& P[A \cup B] + P\left[A \cup B'\right] \\
&= 2P[A] + \left(P[B] + P\left[B'\right]\right) - \left(P[A \cap B] + P\left[A \cap B'\right]\right) \\
0.7 + 0.9 &= 2P[A] + 1 - P[A \cap B] \cup \left(A \cap B'\right) \\
1.6 &= 2P[A] + 1 - P[A] \\
P[A] &= 0.6
\end{aligned}$$

(10)

(d) For $i = 1,\ 2$, let

$R_i =$ event that a red ball is drawn from urn i.

$B_i =$ event that a blue ball is drawn from urn i.

Then if x is the number of blue balls in urn 2.

$$\begin{aligned} 0.44 &= P_r\left[(R_1 \cap R_2) \cup (B_1 \cup B_2)\right] \\ &= P_r\left[R_1 \cap R_2\right] + P_r\left[B_1 \cap B_2\right] \\ &= \frac{4}{10}\left(\frac{16}{x+16}\right) + \frac{6}{10}\left(\frac{x}{x+16}\right) \end{aligned}$$

Therefore

$$\begin{aligned} 2.2 &= \frac{32}{x+16} + \frac{3x}{x+16} \\ &= 3x + \frac{32}{x+16} \\ 2.2x + 35.2 &= 3x + 32 \\ 0.8x &= 3.2 \\ & x + 4 \end{aligned}$$

(10)

[Total 40 marks]

QUESTION 2

(a) Let $N(C)$ denote the number of policy holders in classification C. Then

$$\begin{aligned} & N\left(\text{Young } \cap \text{ Female } \cap \text{ Single}\right) \\ = \;& N(\text{Young } \cap \text{ Female}) - N(\text{Young} \cap -\text{Female} \cap \text{ Married}) \\ = \;& N(\text{Young}) - N(\text{Young} \cap \text{Male}) - \left[N(\text{Young} \cap \text{Married}) - N(\text{Young} \cap \text{Married} \cap \text{Male})\right] \\ = \;& 3000 - 1370 - (1400 - 600) \\ = \;& 880 \end{aligned}$$

(10)

(b) Let

$H =$ event that a death is due to heart disease.

$F =$ event that at least one parent suffered from heart disease.

Then base on the medical records,

$$
\begin{aligned}
P\left(H \cap F^C\right) &= \frac{210-102}{937} = \frac{108}{937} \\
P\left(F^C\right) &= \frac{937-312}{937} = \frac{625}{937} \\
\text{and} \quad P\left[H/F^C\right] &= \frac{P\left[H \cap F^C\right]}{F^C} \\
&= \frac{108}{937} \Big/ \frac{625}{937} \\
&= \frac{108}{625} \\
&= 0.173
\end{aligned}
$$

(10)

(c) Let

$A =$ event that a policyholder has an auto policy.

$H =$ event that a policyholder has a homeowners policy.

Then based on the information given

$$
\begin{aligned}
P_r\left(A \cap H\right) &= 0.15 \\
P_r\left(A \cap H^C\right) &= P_r\left(A\right) - P_r\left(A \cap H\right) \\
&= 0.65 - 0.15 = 0.50 \\
P_r\left(A^C \cap H\right) &= P_r\left(H\right) - P_r\left(A \cap H\right) \\
&= 0.50 - 0.15 = 0.35
\end{aligned}
$$

and the portion of policyholders that will renew at least one policy is given by

$$
\begin{aligned}
& 0.4P_r\left(A \cap H^C\right) + 0.6P_r\left(A^C \cap H\right) + 0.8P_r\left(A \cap H\right) \\
=& \ (0.4)(0.5) + (0.6)(0.35) + (0.8)(0.15) \\
=& \ 0.53 \\
=& \ 53\%
\end{aligned}
$$

(10)

(d) Let

$C =$ event that patient visit a chiropractor.

$T =$ event that patient visit a physical therapist.

We are given that

$$
\begin{aligned}
P_r\left[C\right] &= P_r\left[T\right] + 0.14 \\
P_r\left(C \cap T\right) &= 0.22 \\
P_r\left(C^C \cap T^C\right) &= 0.12
\end{aligned}
$$

Therefore,

$$\begin{aligned} 0.88 &= 1 - P_r\left[C^C \cap T^C\right] \\ &= P_r\left[C \cup T\right] \\ &= P_r\left[C\right] + P_r\left[T\right] - P_r\left[C \cap T\right] \\ &= P_r\left[T\right] + 0.14 + P_r\left[T\right] - 0.22 \\ &= 2P_r\left[T\right] - 0.08 \\ &\text{or} \\ P_r\left[T\right] &= \frac{(0.88 + 0.08)}{2} \\ &= 0.48 \end{aligned}$$

(10)

(e) Let

$M =$ event that customer insures more than one car.

$S =$ event that customer insures a sports car.

Then applying De Morgan's Law, we may compute the desired probability as follows:

$$\begin{aligned} P_r\left(M^C \cap S^C\right) &= P_r\left[(M \cup S)\right]^C \\ &= 1 - P_r\left[(M \cup S)\right] \\ &= 1 - \left[P_r(M) + P_r(S) - P_r(M \cap S)\right] \\ &= 1 - P_r(M) - P_r(S) + P_r(S/M) \\ &= 1 - 0.70 - 0.20 + (0.15)(0.70) \\ &= 0.205 \end{aligned}$$

(5)

(f) Consider the following event about a randomly selected auto insurance customer:

$A =$ customer insures more than one car.

$B =$ customer insures a sport car.

We want to find the probability of the complement of A intersect the complement of B (exactly one car, non – sports).

But $P\left(A^C \cap B^C\right) = 1 - P(A \cup B)$.

And by the Additive Law,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

By the multiplicative Law,

$P(A \cap B) = P(B|A)\,P(A) = 0.15 \times 0.64 = 0.096$

(a) It follows that

$$\begin{aligned} P(A \cup B) &= 0.64 + 0.20 - 0.096 \\ &= 0.744 \text{ and } P\left(A^C \cap B^C\right) = 0.744 \\ &= 0.256 \end{aligned}$$

(10)

[Total 55 marks]

QUESTION 3

(a) If a motor with one or more accidents is regarded as success and $k =$ the number of failures before the fourth success, then k follows a negative binomial distribution and the requested probability is

$$
\begin{aligned}
P_r\,[k > 4] &= 1 - P_r\,[k < 3] \\
&= 1 - \sum_{k=0}^{3} \binom{3+k}{k} \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right)^k \\
&= 1 - \left(\frac{3}{5}\right)^4 \left[\binom{3}{0} \left(\frac{2}{5}\right)^0 + \binom{4}{1} \left(\frac{2}{5}\right)^2 + \binom{6}{3} \left(\frac{2}{5}\right)^3 \right] \\
&= 1 - \left(\frac{3}{5}\right)^4 \left[1 + \frac{8}{5} + \frac{8}{5} + \frac{32}{25} \right] \\
&= 0.2898
\end{aligned}
$$

Alternatively the solution is

$$
\begin{aligned}
& \left(\frac{2}{5}\right)^4 + \binom{4}{1} \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right) + \binom{5}{2} \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^2 + \binom{6}{3} \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^3 \\
= \; & 0.2898
\end{aligned}
$$

which can be desired directly or by regarding the problem as a negative binomial distribution with

(i) success taken as a month no accidents.

(ii) $k =$ the number of failures before the fourth success and

(iii) calculating $P_r\,[k \leq 3]$.

(5)

(b) To determine k, note that

$$
\begin{aligned}
1 &= \int_0^1 k\,(1-y)^4\,dy = -\frac{k}{5}\,(1-y)^5 \Big|_0^1 = \frac{k}{5} \\
k &= 5
\end{aligned}
$$

we next need to find

$$\begin{aligned}
P(V &> 10.000] \\
&= P(100,000Y > 10,000] = P(Y > 0.1] \\
&= \int_{0.1}^{1} 5\,(1-y)^4\,dy \\
&= -\,(1-y)^5\Big|_{0.1}^{1} = (0.9)^5 = 0.59 \\
&\text{and} \quad P\,[Y > 0.1] \\
&= P\,[100.00Y > 40,000] = P\,[Y > 0.4] \\
&= \int_{0.4}^{1} 5\,(1-y)^4\,dy \\
&= (1-y)|_{0.4}^{1} = (0.6)^5 = 0.078
\end{aligned}$$

It is now follows that

$$\begin{aligned}
&P\,[V > 40.000|V > 10.000] \\
=\; & \frac{P\,[V > 40.000 \cap V > 10.000]}{P\,[V > 10.000]} \\
=\; & \frac{P\,[V > 40.000]}{[V > 10.000]} \\
=\; & \frac{0.078}{0.590} \\
=\; & 0.132
\end{aligned}$$

(10)

(c) First note that the density function of X is given by

$$f\,(x) = \begin{cases} \frac{1}{2} & \text{if} \quad 1 = 1 \\ x-1 & \text{if} \quad 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned}
E\,(X) &= \frac{1}{2} + \int_{1}^{2} x\,(x-1)\,dx \\
&= \frac{1}{2} + \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Bigg|_{1}^{2} \\
&= \frac{1}{2} + \frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} \\
&= \frac{7}{3} - 1 \\
&= \frac{4}{3}
\end{aligned}$$

$$
\begin{aligned}
E(X)^2 &= \frac{1}{2} + \int_1^2 x^2 (x-1) + x \\
&= \frac{1}{2} + \int_1^2 \left(x^3 - x^2\right) + x \\
&= \frac{1}{2} + \left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\Big|_1^2 \\
&= \frac{1}{2} + \frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} \\
&= \frac{17}{4} - \frac{7}{3} \\
&= \frac{23}{12}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X)^2 - [E(X)]^2 \\
&= \frac{23}{12} - \left(\frac{4}{3}\right)^2 \\
&= \frac{23}{12} - \frac{16}{9} \\
&= \frac{5}{36}
\end{aligned}
$$

(5)

(d) The marginal density of X is given by

$$
\begin{aligned}
f_x(X) &= \int_0^1 \frac{1}{64}\left(10 - xy^2\right) dy \\
&= \frac{1}{64}\left(10y - \frac{xy^3}{3}\right)\Big|_0^1 \\
&= \frac{1}{64}\left(10 - \frac{x}{3}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
E(X) &= \int_2^{10} x f(x)\, dx \\
&= \int_2^{10} \frac{1}{64}\left(10x - \frac{x^2}{3}\right) dx \\
&= \frac{1}{64}\left(5x^2 - \frac{x^3}{9}\right)\Big|_2^{10} \\
&= \frac{1}{64}\left[\left(500 - \frac{1000}{9}\right) - \left(20 - \frac{8}{9}\right)\right] \\
&= 5.778
\end{aligned}
$$

(5)

[Total 20 marks]