Tutorial letter 201/2/2018

Distribution Theory I

STA1503

Semester 2

Department of Statistics

Solutions to Assignment 1

QUESTION 1

(a)
$$\begin{aligned} P(A) &= 1-P(A) \\ &= 1-0.40 \\ &= 0.60 \end{aligned}$$

(b)
$$\begin{aligned} P(A \cup B) &= P(A)+P(B)-P(A \cap B) \\ &= 0.40+0.25-0.10 \\ &= 0.55 \end{aligned}$$

(c)
$$\begin{aligned} P(A \cup \overline{B}) &= P(A)-P(A \cap B) \\ &= 0.40-0.10 \\ &= 0.30 \end{aligned}$$

(d)
$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= P(\overline{A \cup B}) = 1-P(A \cup B) \\ &= 1-0.55 \\ &= 0.45 \end{aligned}$$

(e)
$$\begin{aligned} P(B/A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{0.10}{0.40} \\ &= 0.25 \end{aligned}$$

(f) Yes, since $P(B/A) = P(B)$ or

Yes, since
$$\begin{aligned} P(A \cap B) &= 0.10 = (0.40)(0.25) \\ &= P(A).P(B) \end{aligned}$$

QUESTION 2

(a) Let Y = time until the first repair, so $Y \sim Exp\ (\mu = 3)$. Thus its CDF is $F(y) = 1-e^{-\frac{y}{3}}, y > 0$, and its survival function is $S(y) = e^{-\frac{y}{3}}$ for

$$\begin{aligned} P(Y > 5 &= S(5) \\ &= e^{-\frac{y}{3}} \\ &= 0.1889 \end{aligned}$$

(b) Let $Y =$ the number of imperfections in $5yds$ of fabric. Then Y~ Poisson $(5.1 = 5)$.

Since there is an average of 1 per yard. Thus

$$\begin{aligned} P(Y \leq 2) &= P(0) + P(1) + P(2) \\ &= \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} + \frac{5^2 e^{-5}}{2!} + \\ &= e^{-5}(1 + 5 + \frac{25}{2}) \\ &= \frac{37}{2} e^{-5} \\ &= 0.1247 \end{aligned}$$

QUESTION 3

(a)

$$\begin{aligned} P(Y \leq 25 \backslash Y > 15) &= \frac{P(15 < Y \leq 25)}{P(Y > 15)} \\ &= \frac{\frac{25-15}{30-0}}{\frac{30-15}{30-0}} \\ &= \frac{25-15}{30-15} \\ &= \frac{10}{15} \\ &= \frac{2}{3} \end{aligned}$$

(b) (i) Choosing answers by tossing a die makes the $n = 8$ trials independent, and since the die is balanced and 2 of the faces yield.

The correct answer each toss, the probability of correct answer each time if $\frac{1}{3}$.Independent trials and constant p

implies a binomial distribution.

(ii) $P(Y = 2) = \binom{8}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^6$

(ii)
$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y < 1) \\ &= 1 - P(Y = 0) \\ &= 1 - \binom{8}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^8 \\ &= 1 - \left(\frac{2}{3}\right)^8 \\ &= 0.961 \end{aligned}$$

QUESTION 3

(a) Let $Y =$ the number of rolls required to obtain a 6.

Then $Y\sim$ Geometric $\left(p = \frac{1}{6}, q = \frac{5}{6}\right)$

$$\begin{aligned} P(Y \leq 3) &= 1 - P(Y > 3) \\ &= 1 - q^3 \\ &= 1 - \left(\frac{5}{6}\right)^3 \\ &= 0.421 \end{aligned}$$

alternatively

$$\begin{aligned} P(Y \leq 3) &= P(Y = 1) + P(Y = 2) + P(Y = 3) \\ &= \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 \\ &= 0.421 \end{aligned}$$

(b) We need to find the smallest positive integer n so that $P(Y \leq n) \geq 0.95$

$$
\begin{aligned}
0.95 &\leq P(Y \leq n) \\
&= 1 - P(Y > n) \\
&= 1 - \left(\frac{5}{6}\right)^n \\
&\Rightarrow \left(\frac{5}{6}\right)^n \leq 1 - 0.95 \\
&= 0.05 \\
n &\Longrightarrow n \ln\left(\frac{5}{6}\right) \leq \ln(0.05) \\
n &\geq \frac{\ln(0.05)}{\ln\left(\frac{5}{6}\right)} \\
n &\geq 16.43
\end{aligned}
$$

Thus we require at least $n = 17$ rolls so that the probability of getting a 6 is at least 0.95

QUESTION 4

(a) Let $Y =$ number of rolls required to obtain a 6. Then $Y\sim$ Geometric $\left(P = \frac{1}{6},\ q = \frac{5}{6}\right)$

$$
\begin{aligned}
P(Y \leq 3) &= 1 - P(Y > 3) \\
&= 1 - q^3 \\
&= 1 - \left(\frac{5}{6}\right)^3 \\
&= 0.421
\end{aligned}
$$

alternatively

$$
\begin{aligned}
P(Y \leq 3) &= P(Y = 1) + P(Y = 2) + P(Y = 3) \\
&= \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 \\
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Thus we require at least $n = 17$ rolls so that the probability of getting a 6 is at least 0.95

QUESTION 5

(a) $P(4 \leq Y \leq 7)$

$$
\begin{aligned}
P(4 \leq Y \leq 7) &= P\left(\frac{4-5}{10} \leq Z \leq \frac{7-5}{10}\right) \\
&= P(-0.1 \leq Z \leq 0.2) \\
\text{Using Text table} &= 1 - P(Z > 0.2) - P(Z > 0.1) \\
&= (1 - 0.4207) - 0.4602) \\
&= 0.1191
\end{aligned}
$$

(b) $P(Y \leq c) = 0.025$

$$\begin{aligned}\Rightarrow P(Z \geq -c) = 0.025 \Rightarrow P\left(\frac{c+5}{10}\right) &= 0.025\\ \text{Using Text table} \quad \Rightarrow \left(\frac{c+5}{10}\right) &= 1.96\\ -c &= 10(1.96) - 5 = 14.6\\ c &= -14.6\end{aligned}$$

QUESTION 6

(a)

$$\begin{aligned}E(\cup_1) &= E(-3Y_1 + 5)\\ &= -E(3Y_1) + 5)\\ &= -3(2) + 5)\\ &= -1\end{aligned}$$

$$\begin{aligned}V(\cup_1) &= V(-3Y_1 + 5)\\ &= V(-3Y_1)\\ &= (-3)^2 V(Y_1)\\ &= 9(5) = 45\end{aligned}$$

(b)

$$\begin{aligned}E(\cup_2) &= V(Y_2 + E(Y_2)]^2\\ &= 9 + (3)^2\\ &= 18\end{aligned}$$

(c)

$$\begin{aligned}E(\cup_3) &= E(7Y_1 - 4Y_2)\\ &= 7(2) - 4(3)\\ &= 2\end{aligned}$$

$$\begin{aligned}V(\cup_3) &= V(7Y_1 - 4Y_2)\\ &= 7^2 V(Y_1) + (-4)^2 V(Y_2) + 2(7)(-4)Cov(Y_1, Y_2)\\ &= 49(5) + 16(9) + 2(7)(-4)(-2)\\ &= 501\end{aligned}$$

QUESTION 7

(a) Differentiating $F(y)$ with respect to y, we have

$$f(y) = \begin{cases} 0 & y \leq 0\\ .125 & 0 \leq y < 2\\ .125 & 2 \leq y < 4\\ 0 & y \geq 4\end{cases}$$

Differentiating $F(y)$ with respect to y, we have

(b) $F(3) - F(1) = \frac{7}{16}$

(c) $1 - F(1.5) = \frac{13}{16}$

(d) $\frac{7}{16}\left(\frac{9}{16}\right) = \frac{7}{9}$.