Tutorial Letter 201/1/2015

Time Series III

STA3704

Semester 1

Department of Statistics

Solutions to Assignment 1

BAR CODE

Question 1

(a) (i) A time series is an ordered sequence of observations observed at equal intervals of time or space.

(ii) Stochastic process is a model that describes the probability structure of a sequence of observations

(iii) Covariance stationarity in a time series occurs when there is a constant mean μ, constant variance σ_t^2, and trendless autocovariance.

(iv) This is a uniqueness of the parameters in a linear process (time series model), whereby the roots of the polynomial equation (characteristics equation) lies outside the unit circle.

(v) A stochastic process Z_t is zero to be strictly stationary if, for all choices of time points $t_1, t_2...t_n$ and all choices of time lag k, the joint distribution of $Z_{t_1}, Z_{t_2}...Z_{t_n}$ is the same as the joint distribution of $Z_{t_{1-k}}, Z_{t_{2-k}}...Z_{t_{n-k}}$.

(vi) A process $\{a_t\}$ is a white noise process if it is a sequence of uncorrelated random variables from a fixed distribution with constant mean $E(a_t) = \mu_a = 0$, constant variance $Var(a_t) = \sigma_a^2$ and $Cov(a_t, a_{t+k}) = 0$ for all $k \neq 0$.

(vii) This is a method used to stationarize homogeneous nonstationary models. It is used to remove trend from a stochastic process (2 marks each = 14)

(b) MA processes are useful in describing phenomena in which events produce an immediate effect that lasts for short periods of time, while the AR processes are useful in describing situations in which the present value of a time series depends on its proceeding values plus a random shock. (6 marks)

Question 2

(a) (i) The process is not stationary because the mean, $E(Z_t)$,and the variance $Var(Z_t)$ varies with (time) t. (3 marks)

(ii) Differencing twice imply: $Y_t = (1-B)^2 X_t \quad = (1-2B+B^2)X_t = X_t - 2BX_t + B^2X_t = X_t - 2X_{t-1} + X_{t-2}$

$X_t = \beta_0 + \beta_1 t + \beta_2 t^2 + a_t$

$X_{t-1} = \beta_0 + \beta_1(t-1) + \beta_2(t-1)^2 + a_{t-1}$

$X_{t-2} = \beta_0 + \beta_1(t-2) + \beta_2(t-2)^2 + a_{t-2}$

Substituting the above in Y_t, we have

$Y_t = X_t - 2X_{t-1} + X_{t-2}$

$\therefore Y_t = a_t - 2a_{t-1} + a_{t-2} + 2\beta_2$, which is free of t.

Testing for stationarity

$$\begin{aligned} E(Y_t) &= 2\beta_2 \\ V(Y_t) &= V\left(a_t - 2a_{t-1} + a_{t-2} + 2\beta_2\right) \\ &= \sigma_a^2(1+4+1) = 6\sigma_a^2 \end{aligned}$$

$\therefore$ Since the mean $E(Y_t)$ and the variance $Var(Y_t)$ are both constants, Y_t is stationary. (Shown) (6 marks)

(b) Process $X_t = \phi_1 X_{t-1} + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}$.

(i) The process/model is $ARMA(1,2)$ (2 marks)

(ii) The Characteristic equation for the AR component is

$$1 - \phi_1 B = 0$$

Solving, we have, $B = 1/\phi_1$, which will be greater than 1 (and thus stationary), provided $\left|\phi_1\right| < 1$ (3 marks)

(iii) Conditions on which θ_1 and θ_2 that makes the process invertible.

$\theta_1 + \theta_2 < 1$

$\theta_2 - \theta_1 < 1$

$|\theta_2| < 1$ (4 marks)

Question 3

(a) (i) The ACF of $Y_t = 10 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}$

$Var(Y_t) = \gamma_0 = var(10 + e_t - \frac{1}{2}e_{t-1} + \frac{1}{4}e_{t-2}) = [1 + (\frac{1}{2})^2 + (\frac{1}{4})^2]\sigma_e^2 = \frac{21}{16}\sigma_e^2$

$$\begin{aligned}
Cov(Y_t, Y_{t-1}) &= Cov(e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}, e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}) \\
&= Cov(-\tfrac{1}{2}e_{t-1}, e_{t-1}) + Cov(\tfrac{1}{4}e_{t-2}, -\tfrac{1}{2}e_{t-2}) \\
&= [-\tfrac{1}{2} + \tfrac{1}{4}(-\tfrac{1}{2})]\sigma_e^2 = -\tfrac{5}{8}\sigma_e^2 \\
Cov(Y_t, Y_{t-2}) &= Cov(e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}, e_{t-2} - \tfrac{1}{2}e_{t-3} + \tfrac{1}{4}e_{t-4}) \\
&= Cov(\tfrac{1}{4}e_{t-2}, e_{t-2}) = \tfrac{1}{4}\sigma_e^2 \\
Cov(Y_t, Y_{t-3}) &= Cov(e_t - \tfrac{1}{2}e_{t-1} + \tfrac{1}{4}e_{t-2}, e_{t-3} - \tfrac{1}{2}e_{t-4} + \tfrac{1}{4}e_{t-5}) = 0
\end{aligned}$$

$\therefore \rho_k = \frac{\gamma_k}{\gamma_0}$

$$\therefore \rho_k = \begin{cases} 1 & k = 0 \\ \frac{-\frac{5}{8}\sigma_e^2}{\frac{21}{16}\sigma_e^2} = -\frac{10}{21} & k = 1 \\ \frac{\frac{1}{4}\sigma_e^2}{\frac{21}{16}\sigma_e^2} = \frac{4}{21} & k = 2 \\ 0 & k > 2 \end{cases}$$ (8 marks)

(ii) The model in a(i) is MA(2) (2 marks)

(b) (i) $E(y_t)$. Here you need to do repeated substitution of the $lags$ of y_t as follows:

$$\begin{aligned}
y_t &= 1 + 1.3y_{t-1} - 0.4y_{t-2} + u_t \\
y_{t-1} &= 1 + 1.3y_{t-2} - 0.4y_{t-3} + u_{t-1} \\
y_{t-2} &= 1 - 1.3y_{t-3} - 0.4y_{t-4} + y_{t-2} \\
y_{t-3} &= 1 + 1.3y_{t-4} - 0.4y_{t-5} + y_{t-3}, \text{ etc.}
\end{aligned}$$

substituting now, we have

$$\begin{aligned} \therefore \quad y_t &= 1 + 1.3\left(1 + 1.3y_{t-2} - 0.4y_{t-3} + u_{t-1}\right) - 0.4y_{t-2} + u_t \\ &= 2.3 + 1.29y_{t-2} - 0.5y_{t-3} + u_t + 1.3u_{t-1} \\ &= 2.3 + 1.29\left[1 + 1.3y_{t-3} - 0.4y_{t-4} + u_{t-2}\right] \\ &\quad -0.52y_{t-3} + u_t + 1.3u_{t-1} \\ &= 3.59 + 1.157y_{t-3} - 0.516y_{t-4} + u_t + 1.3u_{t-1} \\ &\quad +1.29u_{t-2} \end{aligned}$$

Also substituting y_{t-3}, we have

$$y_t = 4.747 + u_t + 1.3u_{t-1} + 1.29u_{t-2} + 1.157u_{t+3} + \cdots$$

$$\therefore \quad E\left(y_t\right) = 4.747 + 0 + 0 + 0 = 4.747 \quad \text{(constant)}$$

(6 marks)

(ii)

$$\begin{aligned} Var\left(y_t\right) &= 0 + 1 + (1.3)^2 + (1.29)^2 + (1.157)^2 + \cdots \\ &= \text{constant (since } Var\left(u_t\right) = 1) \end{aligned}$$

(4 marks)

(iii) Yes, y_t is stationary since the mean and the variance are constants. (2 marks)

(iv) For $AR\,(2)$ model, we know that

$$\begin{aligned} \rho_1 &= \frac{\phi_1}{1-\phi_2} = \frac{1.3}{1-(-0.4)} = 0.9286 \\ \rho_2 &= \frac{\phi_1^2}{1-\phi_2} + \phi_2 = \frac{(1.3)^2}{1-(-0.4)} + (-0.4) \\ &= 0.8071 \end{aligned}$$

(6 marks)

Question 4

(i) $Y_t = 3 + Y_{t-1} + e_t - 0.7e_{t-1}$

$\nabla Y_t = Y_t - Y_{t-1} = 3 + e_t - 0.7e_{t-1}$

So, $E(\nabla Y_t) = 3$ and $Var(\nabla Y_t) = [1 + (0.7)^2]\sigma_e^2 = 1.49\sigma_e^2$ (6 marks)

(ii) $Y_t = 10 + 1.25Y_{t-1} - 0.25Y_{t-2} + e_t - 0.1e_{t-1}$

$\nabla Y_t = Y_t - Y_{t-1} = 10 + 0.25(Y_{t-1} - Y_{t-2}) + e_t - 0.1e_{t-1}$

The model is stationary and invertible ARIMA(1,1,1) model with $\phi = 0.25, \theta = 0.1$, and $\mu = 10$.

Hence, $E(\nabla Y_t) = \frac{\mu}{1-\phi} = \frac{10}{1-0.25} = \frac{10}{0.75} = \frac{40}{3} = 13.333$

Also from Equation (4.4.4), page 78, Variance of ARMA(1,1) is

$Var(\nabla Y_t) = \frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_e^2 = \frac{1-2(0.25)(0.1)+(0.1)^2}{1-(0.25)^2}\sigma_e^2 = 1.024\sigma_e^2$ (8 marks)

(iii) $Y_t = 5 + 2Y_{t-1} - 1.7Y_{t-2} + 0.7Y_{t-3} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$
Factoring the AR characteristic polynomial we have $1-2x+1.7x^2-0.7x^3 = (1-x)(1-x+0.7x^2)$. This shows that a first difference is needed after which a stationary AR(2) is obtained. Thus, the model may be rewritten as $\nabla Y_t = 5 + \nabla Y_{t-1} - 0.7\nabla Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$. So the model is an ARIMA(2,1,2) with $\phi_1 = 1$, $\phi_2 = -0.75, \theta_1 = 0.5, \theta_2 = -0.25$ and $\mu = 5$.

(10 marks)

TOTAL **[90 marks]**