MAT2611 LINEAR ALGEBRA

October/November 2015

SOLUTIONS

Please note: any **fundamental error** is grounds for **no marks** being awarded for an answer.

For some questions, different methods may be used to obtain a correct answer (unless the question specifies the method to be used). Some questions do not have a unique solution. In both cases, full marks will be awarded for answers which answer the given question and are mathematically correct.

QUESTION 1

This question is a **multiple choice** question and should be answered in the **green answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book.

(1.1) Consider the set (2)

$$X := \{\spadesuit\}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $\mathbf{0} = 0$
2. $\mathbf{0} = \spadesuit$
3. $\mathbf{0} = (0, 0)$
4. $\mathbf{0} = (\spadesuit, \spadesuit)$
5. None of the above.

Answer: 2

(1.2) Which of the following are subspaces of M_{22} with the usual operations ? (2)

A. $\text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a = -d \right\}$

Select from the following:

1. Only A.

2. Only A and B.
3. Only B and C.
4. All of A, B and C.
5. None of the above.

Answer: 4

(1.3) Which of the following sets are linearly independent? (2)

A. $\{ (1,0,1),\ (0,0,0) \}$ in $\mathbb{R}^3$

B. $\{ (1,0,1),\ (0,1,0),\ (1,1,1) \}$ in $\mathbb{R}^3$

C. $\left\{ 1-x,\ 1-x^2 \right\}$ in P_2

Select from the following:

1. Only A.
2. Only B.
3. Only A and B.
4. Only C.
5. None of the above.

Answer: 4

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. None of the above.

Answer: 2

(1.5) Which of the following statements are true: (2)

A. $\dim(\operatorname{span}\{ (1,0,1),\ (0,1,0),\ (1,1,1) \}) = 3$ in $\mathbb{R}^3$

B. $\dim(\{ (1,0,1),\ (0,1,0),\ (1,1,1) \}) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \right) = 1$ in M_{22}

Select from the following:

1. Only A.

2. Only B.
3. Only C.
4. Both A and C.
5. None of the above.

Answer: 3

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Both B and C.
5. None of the above.

Answer: 4

(1.7) Which of the following sets are a basis for the column space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both B and C.
4. All of A, B and C.
5. None of the above.

Answer: 3

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? (2)

1. rank$(A) = 2$, nullity$(A) = 1$.
2. rank$(A) = 2$, nullity$(A) = 0$.
3. rank$(A) = 1$, nullity$(A) = 2$.
4. rank$(A) = 3$, nullity$(A) = 0$.
5. None of the above.

Answer: 1

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QUESTION 2

Consider the vector space P_3.

(2.1) Show that (12)

$$\langle p(x), q(x)\rangle := p_0q_0 + p_1q_1 + p_2q_2 + 3p_3q_3,$$

where

$$p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 \quad \text{and} \quad q(x) = q_0 + q_1x + q_2x^2 + q_3x^3,$$

is an inner product on P_3.

We have for $k \in \mathbb{R}$ and

$$p(x) = p_0 + p_1x + p_2x^2 + p_3x^3, q(x) = q_0 + q_1x + q_2x^2 + q_3x^3, r(x) = r_0 + r_1x + r_2x^2 + r_3x^3$$

1. $\langle p(x), q(x)\rangle = p_0q_0 + p_1q_1 + p_2q_2 + 3p_3q_3 = q_0p_0 + q_1p_1 + q_2p_2 + 3q_3p_3 = \langle q(x), p(x)\rangle$ ✓2
2. $$\begin{aligned}\langle p(x) + r(x), q(x)\rangle &= (p_0 + r_0)q_0 + (p_1 + r_1)q_1 + (p_2 + r_2)q_2 + 3(p_3 + r_3)q_3 \\ &= p_0q_0 + r_0q_0 + p_1q_1 + r_1q_1 + p_2q_2 + r_2q_2 + 3p_3q_3 + 3r_3q_3 \\ &= p_0q_0 + p_1q_1 + p_2q_2 + 3p_3q_3 + r_0q_0 + r_1q_1 + r_2q_2 + 3r_3q_3 \\ &= \langle p(x), q(x)\rangle + \langle r(x), q(x)\rangle\end{aligned}$$ ✓4
3. $\langle kp(x), q(x)\rangle = (kp_0)q_0 + (kp_1)q_1 + (kp_2)q_2 + 3(kp_3)q_3 = k(p_0q_0 + p_1q_1 + p_2q_2 + 3p_3q_3) = k\langle p(x), q(x)\rangle$ ✓2
4. $\langle p(x), p(x)\rangle = p_0^2 + p_1^2 + p_2^2 + 3p_3^2 \geq 0$ ✓2 *so that* $\langle p(x), p(x)\rangle \geq 0$ *and* $\langle p(x), p(x)\rangle = 0$ *if and only if* $p_0 = p_1 = p_2 = p_3 = 0$ *(since* $p_0^2, p_1^2, p_2^2, p_3^2 \geq 0$*)* ✓2, *i.e.* $p(x) = 0$.

(2.2) Prove that if $p(x), q(x) \in P_3$, where $p(x), q(x) \neq 0$, are orthogonal to each other with respect to the inner product **defined in 2.1** above, then $\{\, p(x),\ q(x)\,\}$ is a linearly independent set. (6)

Suppose $c_pp(x) + c_qq(x) = 0$ *where* $c_p, c_q \in \mathbb{R}$. *Since* $p(x)$ *and* $q(x)$ *are orthogonal to each other we have* $\langle p(x), q(x)\rangle = \langle q(x), p(x)\rangle = 0$.✓

$$\begin{aligned}c_pp(x) + c_qq(x) = 0 &\Rightarrow \langle p(x), c_pp(x) + c_qq(x)\rangle = \langle p(x), 0\rangle ✓ \\ &\Rightarrow c_p\langle p(x), p(x)\rangle + c_q\langle p(x), q(x)\rangle = 0 ✓ \\ &\Rightarrow c_p\langle p(x), p(x)\rangle = 0 ✓ \\ &\Rightarrow c_p = 0 ✓\end{aligned}$$

since $\langle p(x), p(x)\rangle \neq 0$. *Similarly*

$$\begin{aligned}c_pp(x) + c_qq(x) = 0 &\Rightarrow \langle q(x), c_pp(x) + c_qq(x)\rangle = \langle q(x), 0\rangle \\ &\Rightarrow c_q = 0. ✓\end{aligned}$$

Thus $\{ p(x), q(x) \}$ *is a linearly independent set.*

(2.3) Apply the Gram-Schmidt process to the following subset of P_3: (12)

$$\{ 1 + x^3, \ -1 + x^3, \ -1 + x + x^3 \}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

Let

$$u_1(x) := 1 + x^3, \quad u_2(x) := -1 + x^3, \quad u_3(x) := -1 + x + x^3.$$

Then the Gram-Schmidt process provides

$$\begin{aligned}
v_1(x) &:= u_1 = 1 + x^3 \checkmark \\
\langle v_1(x), v_1(x) \rangle &= 1^2 + 0^2 + 0^2 + 3 \cdot 1^2 = 4 \checkmark \\
\langle u_2(x), v_1(x) \rangle &= -1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 + 3 \cdot 1 \cdot 1 = 2 \checkmark \\
v_2(x) &:= u_2(x) - \frac{\langle u_2(x), v_1(x) \rangle}{\langle v_1(x), v_1(x) \rangle} v_1(x) \checkmark \\
&= (-1 + x^3) - \frac{2}{4}(1 + x^3) = \frac{1}{2}(-3 + x^3) \checkmark \\
\langle v_2(x), v_2(x) \rangle &= \frac{1}{4}((-3)^2 + 0^2 + 0 \cdot 0 + 3 \cdot 1^2) = 3 \checkmark \\
\langle u_3(x), v_1(x) \rangle &= -1 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + 3 \cdot 1 \cdot 1 = 2 \checkmark \\
\langle u_3(x), v_2(x) \rangle &= -1 \cdot \left(-\frac{3}{2} \right) + 1 \cdot 0 + 0 \cdot 0 + 3 \cdot 1 \cdot \frac{1}{2} = 3 \checkmark \\
v_3(x) &:= u_3(x) - \frac{\langle u_3(x), v_1(x) \rangle}{\langle v_1(x), v_1(x) \rangle} v_1 - \frac{\langle u_3(x), v_2(x) \rangle}{\langle v_2(x), v_2(x) \rangle} v_2 \checkmark \\
&= (-1 + x + x^3) - \frac{2}{4}(1 + x^3) - \frac{3}{3} \cdot \frac{1}{2}(-3 + x^3) \\
&= x. \checkmark
\end{aligned}$$

Thus we have the orthogonal basis

$$\left\{ 1 + x^3, \ \frac{1}{2}(-3 + x^3), \ x \right\}. \checkmark^2$$

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QUESTION 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

Row reduction of A *yields*

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_1)$$

which is in upper triangular form, with one nonzero row. Hence the rank is 1, and the nullity is $3-1=2.\checkmark^2$

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda^2(\lambda-2)=0.$$

The characteristic equation is

$$\det\left(\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}\right)\checkmark=\begin{vmatrix}\lambda-1&0&-1\\0&\lambda&0\\-1&0&\lambda-1\end{vmatrix}$$

$$=\lambda((\lambda-1)^2-1)=\lambda((\lambda-1)-1)((\lambda-1)+1)$$

$$=\lambda^2(\lambda-2)=0\checkmark^2.$$

(3.3) Find bases for the eigenspaces of A. (18)

From the characteristic equation we obtain the eigenvalues 0 (twice), and 2$\checkmark^2$. *For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix}-1&0&-1\\0&0&0\\-1&0&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\checkmark^2$$

for $x,y,z\in\mathbb{R}$. *Clearly* $z=-x$. *We find the 2-dimensional eigenspace*

$$\left\{\begin{bmatrix}x\\y\\-x\end{bmatrix}\;:\;x,y\in\mathbb{R}\right\}=\left\{x\begin{bmatrix}1\\0\\-1\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}\;:\;z\in\mathbb{R}\right\}.\checkmark^4$$

Thus a basis is given by

$$\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.\checkmark^2$$

For the eigenspace corresponding to the eigenvalue 2 we solve

$$\begin{bmatrix}1&0&-1\\0&2&0\\-1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\checkmark^2$$

for $x,y,z\in\mathbb{R}$. *Row reduction yields*

$$\begin{bmatrix}1&0&-1\\0&2&0\\-1&0&1\end{bmatrix}\to\begin{bmatrix}1&0&-1\\0&2&0\\0&0&0\end{bmatrix}\qquad(R_3\leftarrow R_3+R_1)$$

so that $y=0$ *and* $x=z$. *The corresponding eigenspace is*

$$\left\{\begin{bmatrix}x\\0\\x\end{bmatrix}\;:\;x\in\mathbb{R}\right\}=\left\{x\begin{bmatrix}1\\0\\1\end{bmatrix}\;:\;z\in\mathbb{R}\right\}.\checkmark^4$$

Thus a basis is given by

$$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}.\checkmark^2$$

(3.4) Prove or disprove: (2)

If B is a 2×2 non-singular matrix, then B is diagonalizable.

The statement is false, for example the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *is non-singular but not diagonalizable.*✓2

(3.5) Let B be an $n \times n$ non-singular matrix. Prove that: (5)

B is diagonalizable if and only if B^{-1} is diagonalizable.

Assume B is diagonalizable, then there exists an invertible $n \times n$ matrix P such that $P^{-1}BP$ is diagonal and non-singular.✓ *It follows that*

$$(P^{-1}BP)^{-1} = P^{-1}B^{-1}(P^{-1})^{-1} = P^{-1}B^{-1}P✓$$

is diagonal (since the inverse of a non-singular diagonal matrix is diagonal).✓ *Hence B^{-1} is diagonalizable.*✓

Similarly, if B^{-1} is diagonalizable then there exists invertible Q such that $Q^{-1}B^{-1}Q$ is diagonal. Then $(Q^{-1}B^{-1}Q)^{-1} = Q^{-1}BQ$ is diagonal.✓

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QUESTION 4

Let $T : M_{22} \to M_{22}$ be defined by $T(A) = A + A^T$ where A^T is the transpose of A.

(4.1) Show that T is a linear transformation. (4)

Let $k \in \mathbb{R}$ and $A, B \in M_{22}$. Using the definition of M_{22} and T we find

- $T(A+B) = (A+B)+(A+B)^T = A+B+A^T+B^T = (A+A^T)+(B+B^T) = T(A)+T(B)$.✓2
- $T(kA) = (kA) + (kA)^T = kA + kA^T = k(A + A^T) = kT(A)$.✓2

(4.2) Find the matrix representation $[T]_B$ of T relative to the basis (10)

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22} ordered from left to right.

From

$$T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ✓2$$

$$T\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ✓2$$

$$T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ✓2$$

$$T\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ✓2$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. ✓2$$

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

The range of T is

$$\begin{aligned} R(T) &= \left\{ T \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \checkmark^2 \\ &= \left\{ \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} \checkmark \qquad (\text{setting } \alpha = 2a,\ \beta = b+c \text{ and } \gamma = 2d) \end{aligned}$$

which is the vector space of 2×2 symmetric matrices. Since $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in M_{22}$ but $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \notin R(T)$, T is not onto.✓

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

$$\begin{aligned} \ker(T) &= \left\{ A \in M_{22} : A + A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \{ A \in M_{22} : A^T = -A \} \checkmark^2 \end{aligned}$$

i.e. the vector space of 2×2 skew-symmetric matrices. Since

$$\begin{aligned} \ker(T) &= \{ A \in M_{22} : A^T = -A \} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ a = 0, b = -c, d = 0 \right\} \\ &= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R}, \right\} \\ &= \operatorname{span} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \end{aligned}$$

we have a one-dimensional space and the nullity of T is 1. ✓2

(4.5) Is T one-to-one? Motivate your answer. (2)

No, since (for example)

$$T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \checkmark^2$$

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TOTAL MARKS: [100]