FORECASTING, TIME SERIES, AND REGRESSION

FOURTH EDITION

BOWERMAN - O'CONNELL - KOEHLER

which implies an *increased* seasonal swing of 964.91 − 569.943 = 394.967 (hundreds of cases) above 569.943, the estimated trend. In general, then, the forecasting equation is appropriate for forecasting a time series with a seasonal swing that is proportional to the average level of the time series as determined by the trend—that is, a time series exhibiting increasing seasonal variation. In fact, sometimes increasing seasonal variation is referred to as **multiplicative seasonal variation.**

7.2 ADDITIVE DECOMPOSITION

Consider a time series that exhibits constant seasonal variation. When the parameters describing the series are not changing over time, the time series can sometimes be modeled adequately by using what is called the **additive decomposition model.**

> The **additive decomposition model** is
>
> $$y_t = TR_t + SN_t + CL_t + IR_t$$

Here TR_t, SN_t, CL_t, and IR_t are again defined to be, respectively, trend, seasonal, cyclical, and irregular factors. However, in this case these factors are additive rather than multiplicative.

The **additive decomposition method** can be used to obtain point estimates tr_t, sn_t, cl_t, and ir_t of the above factors. The procedure begins with the calculation of centered moving averages, CMA_t. The centered moving average is regarded as an estimate of $TR_t + CL_t$. Since the model

$$y_t = TR_t + SN_t + CL_t + IR_t$$

implies that

$$SN_t + IR_t = y_t - (TR_t + CL_t)$$

it follows that the estimate $sn_t + ir_t$ of $SN_t + IR_t$ is

$$sn_t + ir_t = y_t - (tr_t + cl_t) = y_t - CMA_t$$

In order to obtain sn_t, we group the values of $sn_t + ir_t$ by like seasons (months, quarters, etc., as appropriate). For each season we compute the average $\overline{sn}_t$ of the $sn_t + ir_t$ values for that season. We obtain seasonal factors by normalizing the $\overline{sn}_t$ values so that the normalized values sum to zero. The normalization is accomplished by subtracting the quantity $\Sigma_{t=1}^{L}\overline{sn}_t/L$ from each of the $\overline{sn}_t$ values. That is, the estimate of SN_t is

$$sn_t = \overline{sn}_t - \left(\sum_{t=1}^{L} \overline{sn}_t/L\right)$$

We next calculate the deseasonalized observation in time period t to be

$$d_t = y_t - \text{sn}_t$$

Subtracting sn_t from the observation y_t removes the seasonality from the data and allows us to estimate the trend better. We obtain the estimate tr_t of the trend TR_t by fitting a regression equation to the deseasonalized data. For example, a linear trend

$$\text{TR}_t = \beta_0 + \beta_1 t$$

or a quadratic trend

$$\text{TR}_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

might be fitted to the deseasonalized observations.

Since the model

$$y_t = \text{TR}_t + \text{SN}_t + \text{CL}_t + \text{IR}_t$$

implies that

$$\text{CL}_t + \text{IR}_t = y_t - \text{TR}_t - \text{SN}_t$$

it follows that we compute the estimate of $\text{CL}_t + \text{IR}_t$ to be

$$\text{cl}_t + \text{ir}_t = y_t - \text{tr}_t - \text{sn}_t$$

In order to average out ir_t, we compute a three-period moving average of the $\text{cl}_t + \text{ir}_t$ values. That is, the estimate of CL_t is

$$\text{cl}_t = \frac{(\text{cl}_{t-1} + \text{ir}_{t-1}) + (\text{cl}_t + \text{ir}_t) + (\text{cl}_{t+1} + \text{ir}_{t+1})}{3}$$

Finally, we calculate the estimate of IR_t to be

$$\text{ir}_t = (\text{cl}_t + \text{ir}_t) - \text{cl}_t$$

The estimates tr_t, sn_t, cl_t, and ir_t are generally used to describe the time series. We can also use these estimates to compute predictions. If there is no pattern in the irregular component, we predict IR_t to equal zero. It follows that the point forecast of y_t is

$$\hat{y}_t = \text{tr}_t + \text{sn}_t + \text{cl}_t$$

if a well-defined cycle exists and can be predicted. The point forecast is

$$\hat{y}_t = \text{tr}_t + \text{sn}_t$$

if no well-defined cycle exists or if CL_t cannot be predicted. Although there is no theoretically correct prediction interval for y_t, an approximate $100(1 - \alpha)\%$ prediction interval for y_t is

$$[\hat{y}_t \pm B_t[100(1 - \alpha)]]$$

where $B_t[100(1 - \alpha)]$ is the error bound in a $100(1 - \alpha)\%$ prediction interval

$$[\text{tr}_t \pm B_t[100(1 - \alpha)]]$$

for the deseasonalized observation $d_t = y_t - \text{sn}_t$.

SIMPLE EXPONENTIAL SMOOTHING

1. Suppose that the time series $y_1, y_2, \ldots, y_n$ has a level (or mean) that may be slowly changing over time but has no trend or seasonal pattern. Then the estimate ℓ_T for the level (or mean) of the time series in time period T is given by the **smoothing equation**

$$\ell_T = \alpha y_T + (1 - \alpha)\ell_{T-1}$$

where α is a **smoothing constant** between 0 and 1, and ℓ_{T-1} is the estimate of the level (or mean) of the time series in time period $T - 1$.

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = \ell_T \qquad (\tau = 1, 2, 3, \ldots)$$

3. If $\tau = 1$, then a 95% prediction interval computed in time period T for y_{T+1} is

$$[\ell_T \pm z_{[.025]}s]$$

If $\tau = 2$, then a 95% prediction interval computed in time period T for y_{T+2} is

$$\left[\ell_T \pm z_{[.025]}s\sqrt{1 + \alpha^2}\right]$$

In general for any τ, **a 95% prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[\ell_T \pm z_{[.025]}s\sqrt{1 + (\tau - 1)\alpha^2}\right]$$

where the standard error s at time T is

$$s = \sqrt{\frac{SSE}{T-1}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t(t-1)]^2}{T-1}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \ell_{t-1}]^2}{T-1}}$$

Note: There is not general agreement on dividing the SSE by (T – number of smoothing constants). However, we use this divisor because it agrees with the computation of s in the equivalent Box–Jenkins models in Chapters 9 to 12.

EXAMPLE 8.2 In Example 8.1 we saw that $\alpha = .034$ is a "good" value of the smoothing constant when forecasting the 24 observed cod catches in Table 6.1. Therefore, we will use simple exponential smoothing with $\alpha = .034$ to forecast future monthly cod catches. From Figure 8.2(b) we see that $\ell_{24} = 354.5438$ is the estimate made in month 24 of the level (or mean) for the cod catch data. It follows that the point forecast made in month 24 for the cod catch in month 25 and for any other future monthly cod catch is

$$\hat{y}_{24+\tau}(24) = \ell_{24} = 354.5438$$

For the next step, we first need to determine the standard error:

$$
\begin{aligned}
s &= \sqrt{\frac{\sum_{t=1}^{24}[y_t - \ell_{t-1}]^2}{24-1}} \\
&= \sqrt{\frac{(y_1-\ell_0)^2 + (y_2-\ell_1)^2 + \cdots + (y_{24}-\ell_{23})^2}{23}} \\
&= \sqrt{\frac{(362-360.6667)^2 + (381-360.7125)^2 + \cdots + (365-354.1719)^2}{23}} \\
&= \sqrt{\frac{28{,}089.14}{23}} \\
&= 34.95
\end{aligned}
$$

Now, we can compute prediction intervals as follows:

- A 95% prediction interval made in month 24 for y_{25} is

$$
\begin{aligned}
[354.5438 \pm z_{[.025]}s] &= [354.5438 \pm 1.96(34.95)] \\
&= [286.04, 423.05]
\end{aligned}
$$

- A 95% prediction interval made in month 24 for y_{26} is

$$
\begin{aligned}
\left[354.5438 \pm z_{[.025]}s\sqrt{1+\alpha^2}\right] &= \left[354.5438 \pm 1.96(34.95)\sqrt{1+(.034)^2}\right] \\
&= [286.00, 423.09]
\end{aligned}
$$

- A 95% prediction interval made in month 24 for y_{27} is

$$
\begin{aligned}
\left[354.5438 \pm z_{[.025]}s\sqrt{1+2\alpha^2}\right] &= \left[354.5438 \pm 1.96(34.95)\sqrt{1+2(.034)^2}\right] \\
&= [285.96, 423.12]
\end{aligned}
$$

Notice that since the smoothing constant α is small, the increase in the length of the prediction interval is very small.

Now assume that we observe a cod catch in January of year 3 of $y_{25} = 384$. Computers have become so fast with quick access to such a large storage capacity that one could develop a forecasting system, even for thousands of items, that would repeat the process in Example 8.1 to find a new α and s when a new observation is obtained. However, one of the traditional advantages of exponential smoothing is that we need only our last estimate to find new point forecasts. For the cod catch data, we can update ℓ_{24} to ℓ_{25} by using the smoothing equation

$$
\begin{aligned}
\ell_{25} &= \alpha y_{25} + (1-\alpha)\ell_{24} \\
&= .034(384) + .966(354.5438) \\
&= 355.5453
\end{aligned}
$$

This implies that the point forecast made in month 25 of the cod catch in month 26 and of any other future month is

$$\hat{y}_{25+\tau}(25) = \ell_{25} = 355.5453$$

Furthermore, it follows that a 95% prediction interval made in month 25 for y_{26} is

$$[355.5453 \pm z_{[.025]}s] = [355.5453 \pm 1.96(34.95)]$$
$$= [287.04, 424.05]$$

and a 95% prediction interval made in month 25 for y_{27} is

$$\left[355.5453 \pm z_{[.025]}s\sqrt{1+\alpha^2}\right] = \left[355.5453 \pm 1.96(34.95)\sqrt{1+(.034)^2}\right]$$
$$= [287.00, 424.09]$$

In general, note that the smoothing equation

$$\ell_T = \alpha y_T + (1-\alpha)\ell_{T-1}$$

implies

$$\ell_{T-1} = \alpha y_{T-1} + (1-\alpha)\ell_{T-2}$$

Substitution, therefore, gives us

$$\ell_T = \alpha y_T + (1-\alpha)[\alpha y_{T-1} + (1-\alpha)\ell_{T-2}]$$
$$= \alpha y_T + (1-\alpha)\alpha y_{T-1} + (1-\alpha)^2\ell_{T-2}$$

Substituting recursively for $\ell_{T-2}, \ell_{T-3}, \ldots, \ell_1$, and ℓ_0, we obtain

$$\ell_T = \alpha y_T + (1-\alpha)\alpha y_{T-1} + (1-\alpha)^2\alpha y_{T-2} + \cdots + (1-\alpha)^{T-1}\alpha y_1 + (1-\alpha)^T\ell_0$$

The coefficients measuring the contribution of the observations $y_T, y_{T-1}, y_{T-2}, \ldots, y_1$ are $\alpha, (1-\alpha)\alpha, (1-\alpha)^2\alpha, \ldots, (1-\alpha)^{T-1}\alpha$, respectively, and are decreasing exponentially with age. For this reason we refer to this procedure as simple exponential smoothing.

Since the coefficients are decreasing exponentially, the most recent observation y_T makes the largest contribution to the current estimate for the level (or mean). Older observations make smaller and smaller contributions to this estimate. Thus, remote observations are "dampened out" of the current estimate of the level (or mean) as time advances. The rate at which remote observations are dampened out depends on the smoothing constant α. For example, if $\alpha = .9$ we obtain coefficients .9, .09, .009, .0009, ... For values of α near 0, remote observations are dampened out more slowly. The choice of a smoothing constant α is usually made by simulated forecasting of a historical data set as illustrated in Example 8.1.

The smoothing equation may be written in what is called the "error correction form" as follows.

ERROR CORRECTION FORM

The **error correction form** for the smoothing equation in **simple exponential smoothing:**

$$\ell_T = \ell_{T-1} + \alpha(y_T - \ell_{T-1})$$

This form of the smoothing equation says that ℓ_T, the estimate of the level at time period T, is the sum of ℓ_{T-1}, the estimate of the level at time period $T-1$, and a fraction α of $(y_T - \ell_{T-1})$, which is the one-period-ahead forecast error. Thus, at each time period T, we use the new observation y_T to adjust the estimate ℓ_{T-1} upward or downward depending on the size and sign of our forecast error $(y_T - \ell_{T-1})$. We can easily see that the error correction form of the smoothing equation produces the same estimate ℓ_T as the original form of the smoothing equation because

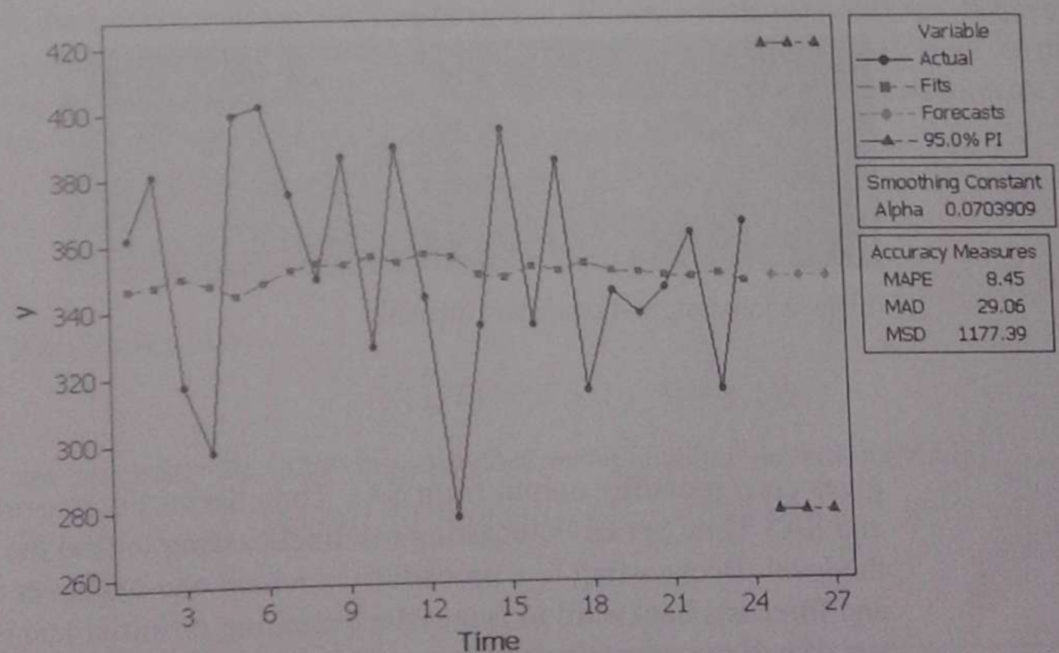
$$\begin{aligned}\ell_T &= \ell_{T-1} + \alpha(y_T - \ell_{T-1}) \\ &= \ell_{T-1} + \alpha y_T - \alpha\ell_{T-1} \\ &= \alpha y_T + (1-\alpha)\ell_{T-1}\end{aligned}$$

The error correction form of the smoothing equation is better suited for understanding the state space model for the simple exponential smoothing method that is introduced in optional Section 8.6.

Many computer software packages for forecasting include exponential smoothing as a choice. These packages choose the initial value(s) and the smoothing constant(s) in different ways and also compute approximate prediction intervals in different ways. The user should carefully investigate how the computer software package implements exponential smoothing.

Figure 8.3 gives the MINITAB output of using simple exponential smoothing to forecast in month 24 the cod catches in the next three months. Figure 8.4

FIGURE 8.3
MINITAB output of optimization results for cod catch data

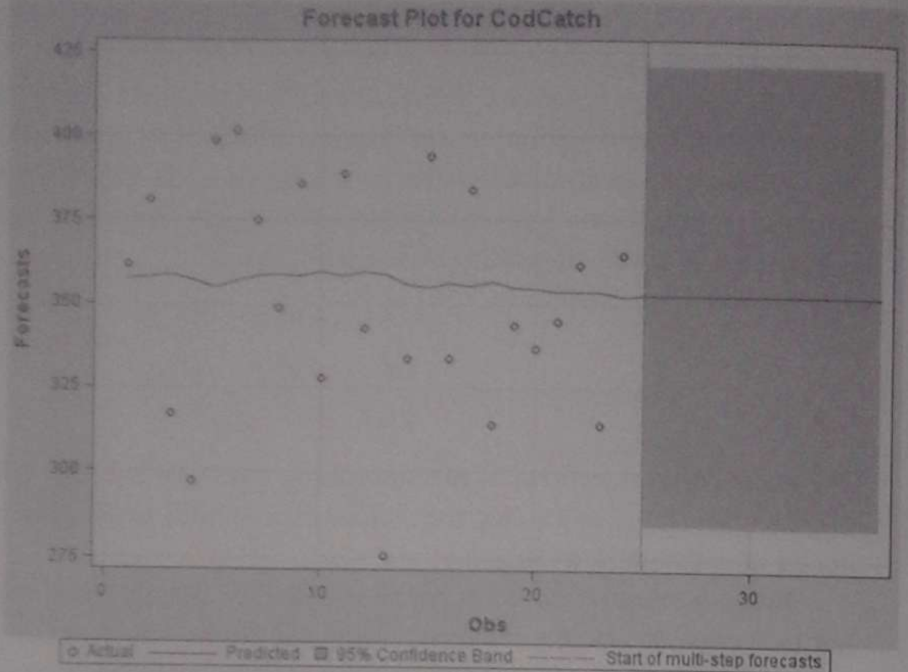


(a) Smoothing constant and graph of forecasts

25	348.168	276.976	419.360
26	348.168	276.976	419.360
27	348.168	276.976	419.360

(b) Point forecasts and prediction intervals

FIGURE 8.4
SAS output of optimization results for cod catch data



Mean Square Error	1150.03403

Parameter	Estimate	Standard Error	t Value	Approx Pr > |t|
Level Weight	0.03817	0.05096	0.75	0.4614

(a) Smoothing constant and graph of forecasts

Obs	Forecasts	Standard Error	95% Confidence Limits	
25	352.8360	34.6415	284.9399	420.7322
26	352.8360	34.6668	284.8904	420.7816
27	352.8360	34.6920	284.8410	420.8310

(b) Point forecasts and prediction intervals

gives corresponding output from SAS Time Series Forecasting. Both MINITAB and SAS Time Series Forecasting use **backcasting** to find the initial estimate for the level. Backcasting is a procedure in which one begins at the end of the data and forecasts backward to forecast a beginning or initial level. We used the average of half the data (12 observations) as our initial estimate ℓ_0 for the level. It can be shown that in simple exponential smoothing, using six observations is reasonable. Generally, one would expect larger smoothing constants to be selected if less data is used to find the initial estimate. The prediction intervals in the SAS Time Series Forecasting output are computed by the same formulas that we used in Example 8.2.

The smoothing constants of .070 from MINITAB and .038 from SAS Time Series Forecasting are both different from our value of .034. However, all three approaches indicate that the level of the cod catch data is not changing much. The point forecasts of 348.2 from MINITAB and 352.8 from SAS Time Series Forecasting also differ from our point forecast of 354.5. Given that there is quite a bit of variation in the data and that the level (or mean) is not changing much, these forecasts are not very different.

8.2 TRACKING SIGNALS

It is sometimes necessary to change the smoothing constant(s) being employed in exponential smoothing. For example, when simple exponential smoothing is being used, the rate at which the level is changing over time could change. It is possible that a different smoothing constant α would produce improved forecasts. A **tracking signal** might help us decide when something is wrong with a forecasting system (for instance, when we are using an inappropriate smoothing constant). Although we know that a forecasting system will never produce perfect forecasts, a tracking signal can tell us when our forecast errors are larger than an "accurate" forecasting system might reasonably be expected to produce.

The first tracking signal was the **simple cusum tracking signal,** which was suggested by R. G. Brown (1959). It is a ratio that compares the cumulative sum of errors to the smoothed mean absolute deviation. To begin with, suppose that we have a history of T single-period-ahead forecast errors, $e_1(\alpha), e_2(\alpha), \ldots, e_T(\alpha)$. Here (α) denotes the particular value of α employed to obtain the single-period-ahead forecast errors. We next define the sum (Y) of these forecast errors:

$$Y(\alpha, T) = \sum_{t=1}^{T} e_t(\alpha)$$

It is obvious that

$$Y(\alpha, T) = Y(\alpha, T-1) + e_T(\alpha)$$

and we define the following smoothed mean absolute deviation (MAD):

$$\text{MAD}(\alpha, T) = \alpha|e_t(\alpha)| + (1-\alpha)\text{MAD}(\alpha, T-1)$$

Then:

The **simple cusum tracking signal** $C(\alpha, T)$ is defined as

$$C(\alpha, T) = \left|\frac{Y(\alpha, T)}{\text{MAD}(\alpha, T)}\right|$$

If $C(\alpha, T)$ is "large" this means that $Y(\alpha, T)$ is large relative to the mean absolute deviation $\text{MAD}(\alpha, T)$. This in turn says that the forecasting system is producing errors that are either consistently positive or consistently negative. That is, a large value of $C(\alpha, T)$ implies that the forecasting system is producing forecasts that are either consistently smaller or consistently larger than the time series values being forecasted. Since an "accurate" forecasting system should be producing roughly one-half positive errors and one-half negative errors, a large value of $C(\alpha, T)$ indicates that the forecasting system is not performing accurately. In practice, if $C(\alpha, T)$ exceeds a control limit, denoted by K, for two or more consecutive periods, this is taken as a strong indication that the forecast errors have been larger than an accurate forecasting system can reasonably be expected to produce.

Initial values must be assigned to $Y(\alpha, T)$ and $\text{MAD}(\alpha, T)$ when starting the forecasting procedure. Since it is reasonable to assume that the original model is correct, $Y(\alpha, 0) = 0$ is the starting value for $Y(\alpha, T)$. Since there is some random variation in the process, however, it is not reasonable to use $\text{MAD}(\alpha, 0) = 0$ as the starting value for $\text{MAD}(\alpha, T)$. One possible initial value for $\text{MAD}(\alpha, 0)$ is the average of the absolute values of the one-step-ahead forecast errors found in the forecasting of the historical data when the optimal smoothing constant α is used.

Values that have been recommended for the control limit K apply to low values of α (no larger than .30). Gardner (1983) used simulations to provide a table of control limits when α = .1, .2, or .3. For control limits that have only a 5% chance of incorrectly indicating a large $C(\alpha, T)$, the values of the control limit K are

α	.1	.2	.3
K	5.6	4.1	3.5

For control limits that have only a 1% chance of incorrectly indicating a large $C(\alpha, T)$, the values of the control limit K are

α	.1	.2	.3
K	7.5	5.6	4.9

Another tracking signal that has had extensive use in practice is the **smoothed error tracking signal,** which was developed by Trigg (1964). It is the ratio of the smoothed one-period-ahead forecasting error to the smoothed mean absolute deviation. We define the smoothed error (E) of the one-period-ahead forecast errors as

$$E(\alpha, T) = \alpha e_1(\alpha) + (1-\alpha)E(\alpha, T-1)$$

Then:

The **smoothed error tracking signal** is defined as

$$S(\alpha, T) = \left| \frac{E(\alpha, T)}{\text{MAD}(\alpha, T)} \right|$$

More information on the pros and cons of these and other tracking signals that have had successful use in practice can be found in Gardner (1983).

Tracking signals no longer play the extensive role they once did in forecasting. With the great speed and storage capacity of today's computers, the smoothing constant(s) in exponential smoothing can be reestimated frequently. It is no longer advantageous, when there is trend in the time series data, to use double exponential smoothing with one parameter instead of Holt's two-parameter smoothing (see Section 8.3). The tracking signals for time series with trend were based on smoothing methods with one parameter. Furthermore, there are new procedures based on new models for the exponential smoothing methods to aid in better identification of the correct smoothing method (see Hyndman et al., 2002).

8.3 HOLT'S TREND CORRECTED EXPONENTIAL SMOOTHING

Suppose that a time series displays a linear trend. If the time series is increasing or decreasing at approximately a fixed rate, then the time series may be described by the linear trend model (see Section 6.1)

$$y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

The **level** (or mean) at time T is $\beta_0 + \beta_1 T$, and the level (or mean) at time $T - 1$ is $\beta_0 + \beta_1(T - 1)$. Thus, the increase or decrease in the level of the time series from time period $T - 1$ to time period T is

$$[\beta_0 + \beta_1 T] - [\beta_0 + \beta_1(T - 1)] = \beta_1$$

This fixed rate of increase or decrease β_1 is called the **growth rate.**

Holt's trend corrected exponential smoothing is appropriate when both the level and the growth rate are changing. A model different from the linear trend model is needed to describe the changing level and growth rate (see optional Section 8.6). To implement Holt's trend corrected exponential smoothing, we let ℓ_{T-1} denote the estimate of the level of the time series in time period $T - 1$, and we let b_{T-1} denote the estimate of the growth rate of the time series in time $T - 1$. Then, if we observe a new time series value y_T in time period T, we use two smoothing equations to update the estimates ℓ_{T-1} and b_{T-1}. The estimate of the level in time period T uses the *smoothing constant* α and is

$$\ell_T = \alpha y_T + (1 - \alpha)[\ell_{T-1} + b_{T-1}]$$

This equation says that ℓ_T equals a fraction α of the newly observed time series value y_T plus a fraction $(1 - \alpha)$ of $[\ell_{T-1} + b_{T-1}]$, which is the estimate of the level of the time series in time period T, as calculated using estimates ℓ_{T-1} and b_{T-1} computed in time period $T - 1$. The estimate of the growth rate of the time series in time period T uses the *smoothing constant* γ and is

$$b_T = \gamma[\ell_T - \ell_{T-1}] + (1 - \gamma)b_{T-1}$$

This equation says that b_T equals a fraction γ of $[\ell_T - \ell_{T-1}]$, which is an estimate of the difference between the levels in periods T and $T-1$, plus a fraction $(1-\gamma)$ of b_{T-1}, the estimate of the growth rate made in time period $T-1$.

We summarize the procedure in the following box.

HOLT'S TREND CORRECTED EXPONENTIAL SMOOTHING

1. Suppose that the time series $y_1, y_2, \ldots, y_n$ exhibits a linear trend for which the level and growth rate may be changing with no seasonal pattern. Then **the estimate ℓ_T for the level of the time series** and **the estimate b_T for the growth rate of the time series** in time period T are given by the smoothing equations

$$\ell_T = \alpha y_T + (1-\alpha)[\ell_{T-1} + b_{T-1}]$$
$$b_T = \gamma[\ell_T - \ell_{T-1}] + (1-\gamma)b_{T-1}$$

where α and γ are smoothing constants between 0 and 1, and ℓ_{T-1} and b_{T-1} are estimates at time $T-1$ for the level and growth rate, respectively.

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = \ell_T + \tau b_T \qquad (\tau = 1, 2, \ldots)$$

3. If $\tau = 1$, then a **95% prediction interval computed in time period T for y_{T+1}** is

$$[(\ell_T + b_T) \pm z_{[.025]}s]$$

If $\tau = 2$, then a 95% prediction interval computed in time period T for y_{T+2} is

$$\left[(\ell_T + 2b_T) \pm z_{[.025]}s\sqrt{1 + \alpha^2(1+\gamma)^2}\right]$$

If $\tau = 3$, then a 95% prediction interval computed in time period T for y_{T+3} is

$$\left[(\ell_T + 3b_T) \pm z_{[.025]}s\sqrt{1 + \alpha^2(1+\gamma)^2 + \alpha^2(1+2\gamma)^2}\right]$$

In general for $\tau \geq 2$, **a 95% prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[(\ell_T + \tau b_T) \pm z_{[.025]}s\sqrt{1 + \sum_{j=1}^{\tau-1} \alpha^2(1+j\gamma)^2}\right]$$

where the standard error s computed in time period T is

$$s = \sqrt{\frac{SSE}{T-2}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t(t-1)]^2}{T-2}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - (\ell_{t-1} + b_{t-1})]^2}{T-2}}$$

EXAMPLE 8.3 In this example, we use Holt's trend corrected exponential smoothing to forecast the weekly thermostat sales time series given in Table 8.1. A plot of the sales data versus time is shown in Figure 8.5. Although the plot of the sales data indicates an upward trend for the sales in the latter weeks, the growth rate of sales has clearly been changing over the 52-week period. Moreover, there is no seasonal pattern. Thus, Holt's trend corrected exponential smoothing is an appropriate forecasting procedure to apply to this time series.

To start the procedure for using the two smoothing equations, we first obtain an initial estimate ℓ_0 for the level and an initial estimate b_0 for the growth rate in time period 0. One way to do this is to fit a least squares trend line to part (say half) of the historical data and let the y-intercept be ℓ_0 and the slope be b_0. For example, consider

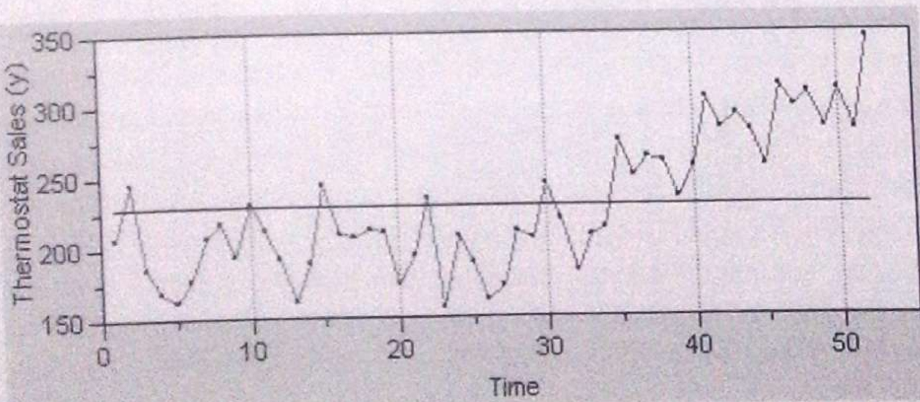
TABLE 8.1 Weekly Thermostat Sales

206	189	172	255
245	244	210	303
185	209	205	282
169	207	244	291
162	211	218	280
177	210	182	255
207	173	206	312
216	194	211	296
193	234	273	307
230	156	248	281
212	206	262	308
192	188	258	280
162	162	233	345

Note: Read downward left to right.

Source: Reprinted from R. G. Brown (1962), *Smoothing, Forecasting, and Prediction of Discrete Time Series*, p. 431, by permission of Prentice-Hall, Inc. Suggested by an example in Abraham and Ledolter (1983).

FIGURE 8.5
JMP IN plot of weekly thermostat sales



the 52 observations of the thermostat sales values in Table 8.1. If we fit a least squares trend line to the first 26 of these values, we obtain

$$\hat{y}_t = 202.6246 - .3682t$$

This would imply that $\ell_0 = 202.6246$ and $b_0 = -.3682$. We have found these two values by performing a simple linear regression analysis in Excel. Figure 8.6 presents an Excel spreadsheet with the results of the regression in the two rightmost columns. We have copied the values from the "Coefficients" column to the first cells in the columns "Level" and "Growth Rate," in the row "Time Period 0."

FIGURE 8.6 Excel spreadsheet of Holt's trend corrected exponential smoothing for thermostat sales, $\alpha = .20$, $\gamma = .10$

	A	B	C	D	E	F	G	H	I
1	n	alpha	gamma	SSE		ssquare	s		
2	52	0.2	0.1	39182		783.65	27.99		
3									
4		Actual			Forecast		Squared		
5	Time	Themostat		Growth	Made Last	Forecast	Forecast		
6	Period	Sales (y)	Level	Rate	Period	Error	Error		
7	0		202.6246	-0.3682				SUMMARY OUTPUT	
8	1	206	203.0051	-0.2933	202.2564	3.7436	14.0145		
9	2	245	211.1694	0.5524	202.7118	42.2882	1788.2923	*Regression Statistics*	
10	3	185	206.3775	0.0180	211.7219	-26.7219	714.0582	Multiple R	0.1117696
11	4	169	198.9164	-0.7299	206.3955	-37.3955	1398.4224	R Square	0.0124924
12	5	162	190.9492	-1.4536	198.1865	-36.1865	1309.4608	Adjusted R S	-0.028654
13	6	177	186.9964	-1.7036	189.4955	-12.4955	156.1383	Standard Err	25.555174
14	7	207	189.6343	-1.2694	185.2929	21.7071	471.1995	Observations	26
15	8	216	193.8919	-0.7167	188.3649	27.6351	763.6997		
16	9	193	193.1402	-0.7202	193.1752	-0.1752	0.0307	ANOVA	
17	10	230	199.9360	0.0314	192.4199	37.5801	1412.2609		*df*
18	11	212	202.3739	0.2720	199.9673	12.0327	144.7850	Regression	1
19	12	192	200.5167	0.0591	202.6459	-10.6459	113.3354	Residual	24
20	13	162	192.8607	-0.7124	200.5758	-38.5758	1488.0961	Total	25
21	14	189	191.5186	-0.7754	192.1483	-3.1483	9.9117		
22	15	244	201.3946	0.2898	190.7433	53.2567	2836.2799		*Coefficients*
23	16	209	203.1475	0.4361	201.6844	7.3156	53.5182	Intercept	202.62462
24	17	207	204.2669	0.5044	203.5836	3.4164	11.6718	X Variable 1	-0.368205
25	18	211	206.0170	0.6290	204.7713	6.2287	38.7969		
53	46	312	290.7142	4.9749	285.3928	26.6072	707.9453		
54	47	296	295.7513	4.9811	295.6891	0.3109	0.0966		
55	48	307	301.9860	5.1065	300.7324	6.2676	39.2823		
56	49	281	301.8740	4.5846	307.0924	-26.0924	680.8155		
57	50	308	306.7669	4.6155	306.4586	1.5414	2.3759		
58	51	280	305.1059	3.9878	311.3823	-31.3823	984.8515		
59	52	345	316.2750	4.7059	309.0937	35.9063	1289.2627		

Starting with ℓ_0 and b_0, we calculate a point forecast of y_1 from time origin 0 to be

$$\hat{y}_1(0) = \ell_0 + b_0 = 202.6246 + (-.3682) = 202.2564$$

This point forecast is shown in the Excel spreadsheet of Figure 8.6 in the column headed "Forecast Made Last Period." Also shown in the spreadsheet are the actual thermostat sales value $y_1 = 206$ and the forecast error, which is

$$y_1 - \hat{y}_1(0) = 206 - 202.2564 = 3.7436$$

We next choose values of the smoothing constants α and γ. A reasonable choice is $\alpha = .2$ and $\gamma = .1$. Then, using $y_1 = 206$ and the equation for ℓ_T, it follows that the estimate of the level of the time series in time period 1 is

$$\begin{aligned} \ell_1 &= \alpha y_1 + (1-\alpha)[\ell_0 + b_0] \\ &= .2(206) + .8[202.6246 + (-.3682)] \\ &= 203.0051 \end{aligned}$$

Furthermore, using the equation for b_T, the estimate for the growth rate of the time series in time period 1 is

$$\begin{aligned} b_1 &= \gamma[\ell_1 - \ell_0] + (1-\gamma)b_0 \\ &= .1[203.0051 - 202.6246] + .9(-.3682) \\ &= -.2933 \end{aligned}$$

It follows that a point forecast made in time period 1 of y_2 is

$$\hat{y}_2(1) = \ell_1 + b_1 = 203.0051 + (-.2933) = 202.7118$$

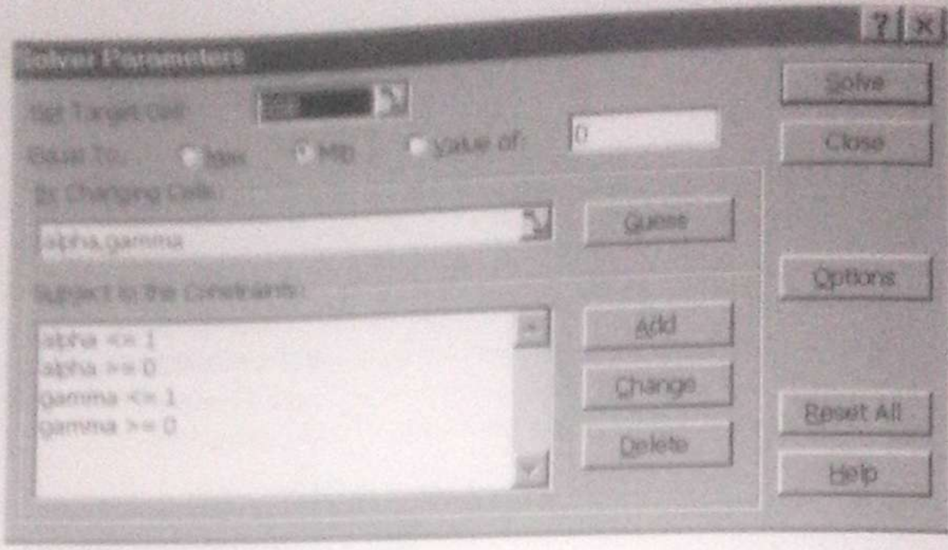
Since the actual thermostat sales value in period 2 is $y_2 = 245$, the forecast error is

$$y_2 - \hat{y}_2(1) = 245 - 202.7118 = 42.2882$$

The Excel spreadsheet in Figure 8.6 shows the entire process of using the two smoothing equations in Holt's trend corrected exponential smoothing to find new period-by-period estimates of the level and growth rate of the entire time series. The spreadsheet also shows the one-period-ahead forecasts, forecast errors, and squared forecast errors. Since the formulas for entries in each time period are the same, we can easily copy the formulas from time period 1 to the other 51 time periods to produce the spreadsheet in Figure 8.6. The spreadsheet also shows that the sum of the squared forecast errors (SSE) is 39,182 when $\alpha = .2$ and $\gamma = .1$.

To find "good" values for α and γ, we can use Solver in Excel to find the values for α and γ that produce a minimum value for the SSE. Figure 8.7(a) presents the setup in the Solver dialog box for finding the values of α and γ that minimize the SSE when possible values of α and γ range from 0 to 1. Figure 8.7(b) gives the Excel spreadsheet of all the values when the SSE is at its minimum value of 38,884. When the SSE is minimized, $\alpha = .247$, $\gamma = .095$, $s = 27.89$, $\ell_{52} = 315.9460$, and $b_{52} = 4.5040$. We can now use these values to find forecasts and 95% prediction intervals for future thermostat sales. One note of caution is that sometimes the SSE has not reached a minimum value because it is stuck at a local minimum value. It is advisable to enter different values for the smoothing constants

FIGURE 8.7 Finding the values of α and γ that minimize SSE in Holt's trend corrected exponential smoothing for thermostat sales



(a) Solver in Excel for finding α (alpha) and γ (gamma) that minimize the SSE

	A	B	C	D	E	F	G
1	n	alpha	gamma	SSE		ssquare	s
2	52	0.247	0.095	38884		777.68	27.89
3							
4		Actual			Forecast		Squared
5	Time	Themostat		Growth	Made Last	Forecast	Forecast
6	Period	Sales (y)	Level	Rate	Period	Error	Error
7	0		202.6246	-0.3682			
8	1	206	203.1805	-0.2804	202.2564	3.7436	14.0145
9	2	245	213.2921	0.7074	202.9001	42.0999	1772.4001
10	3	185	206.8413	0.0270	213.9996	-28.9996	840.9751
							⋮
56	49	281	301.0910	4.2475	307.6757	-26.6757	711.5940
57	50	308	305.9955	4.3100	305.3386	2.6614	7.0833
58	51	280	302.8248	3.5989	310.3055	-30.3055	918.4226
59	52	345	315.9460	4.5040	306.4237	38.5763	1488.1288

(b) Excel spreadsheet with the values for α, γ, s, ℓ_{52}, and b_{52} when SSE is minimized

in Figure 8.6 and reoptimize (that is, run Solver with different values) to see if a minimum SSE has been found in Figure 8.7.

To illustrate the forecasting of thermostat sales, we use $\ell_{52} = 315.9460$ and $b_{52} = 4.5040$ to find the point forecasts for y_{53}, y_{54}, and y_{55} as follows:

$$\hat{y}_{53}(52) = \ell_{52} + b_{52} = 315.9460 + 4.5040 = 320.45$$

$$\hat{y}_{54}(52) = \ell_{52} + 2b_{52} = 315.9460 + 2(4.5040) = 324.954$$

$$\hat{y}_{55}(52) = \ell_{52} + 3b_{52} = 315.9460 + 3(4.5040) = 329.458$$

In order to compute the 95% prediction intervals, we use the forecast errors in Figure 8.7(b) to compute the standard error s. This figure gives the one-period-ahead forecast errors when we use $\alpha = .0247$ and $\gamma = .095$, the smoothing constants that minimize the SSE.

We obtain

$$
\begin{aligned}
s &= \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t(t-1)]^2}{T-2}} \\
&= \sqrt{\frac{(3.7436)^2 + (42.0999)^2 + \cdots + (38.5763)^2}{50}} \\
&= \sqrt{\frac{38,884}{50}} \\
&= 27.89
\end{aligned}
$$

Then, a 95% prediction interval for y_{53} is

$$
\begin{aligned}
[\hat{y}_{53}(52) \pm z_{[.025]}s] &= [(\ell_{52} + b_{52}) \pm z_{[.025]}s] \\
&= [320.45 \pm 1.96(27.89)] \\
&= [320.45 \pm 54.66] \\
&= [265.79,\ 375.11]
\end{aligned}
$$

a 95% prediction interval for y_{54} is

$$
\begin{aligned}
&\left[\hat{y}_{54}(52) \pm z_{[.025]}s\sqrt{1+\alpha^2(1+\gamma)^2}\right] \\
&= \left[(\ell_{52} + 2b_{52}) \pm z_{[.025]}s\sqrt{1+\alpha^2(1+\gamma)^2}\right] \\
&= \left[324.954 \pm 1.96(27.89)\sqrt{1+(.247)^2(1+.095)^2}\right] \\
&= [324.954 \pm 56.63] \\
&= [268.32,\ 381.58]
\end{aligned}
$$

and a 95% prediction interval for y_{55} is

$$
\begin{aligned}
&\left[\hat{y}_{55}(52) \pm z_{[.025]}s\sqrt{1+\alpha^2(1+\gamma)^2+\alpha^2(1+2\gamma)^2}\right] \\
&= \left[(\ell_{52} + 3b_{52}) \pm z_{[.025]}s\sqrt{1+\alpha^2(1+\gamma)^2+\alpha^2(1+2\gamma)^2}\right] \\
&= \left[329.458 \pm 1.96(27.89)\sqrt{1+(.247)^2(1+.095)^2+(.247)^2(1+2(.095))^2}\right] \\
&= [329.458 \pm 58.86] \\
&= [270.60,\ 388.32]
\end{aligned}
$$

Furthermore, if we observe $y_{53} = 330$, we can either find a new optimal α and γ that minimize the SSE for 53 time periods and compute a new s, or we can simply revise the estimate for the level and growth rate and our forecasts as follows:

$$
\begin{aligned}
\ell_{53} &= \alpha y_{53} + (1-\alpha)[\ell_{52} + b_{52}] \\
&= .247(330) + .753[315.946 + 4.5040] \\
&= 322.8089 \\
b_{53} &= \gamma[\ell_{53} - \ell_{52}] + (1-\gamma)b_{52} \\
&= .095[322.8089 - 315.9460] + .905(4.5040) \\
&= 4.7281
\end{aligned}
$$

Then the revised point forecasts for y_{54} and y_{55} are

$$\hat{y}_{54}(53) = \ell_{53} + b_{53} = 322.8089 + 4.7281 = 327.537$$

$$\hat{y}_{55}(53) = \ell_{53} + 2b_{53} = 322.8089 + 2(4.7281) = 332.2651$$

and the 95% prediction intervals for y_{54} and y_{55} are

$$\begin{aligned}[\hat{y}_{54}(53) \pm z_{[.025]}s] &= [327.537 \pm 1.96(27.89)] \\ &= [327.537 \pm 54.66] \\ &= [272.88,\ 382.20]\end{aligned}$$

$$\begin{aligned}\left[\hat{y}_{55}(53) \pm z_{[.025]}s\sqrt{1+\alpha^2(1+\gamma)^2}\right] &= \left[332.2651 \pm 1.96(27.89)\sqrt{1+(.247)^2(1+.095)^2}\right] \\ &= [332.2651 \pm 56.63] \\ &= [275.64,\ 388.90]\end{aligned}$$

The smoothing equations for Holt's trend corrected exponential smoothing can also be put in the error correction form. As in simple exponential smoothing, the error correction form is easier to relate to the models in optional Section 8.6.

ERROR CORRECTION FORM

The error correction form for the smoothing equations in **Holt's trend corrected exponential smoothing:**

$$\ell_T = \ell_{T-1} + b_{T-1} + \alpha[y_T - (\ell_{T-1} + b_{T-1})]$$

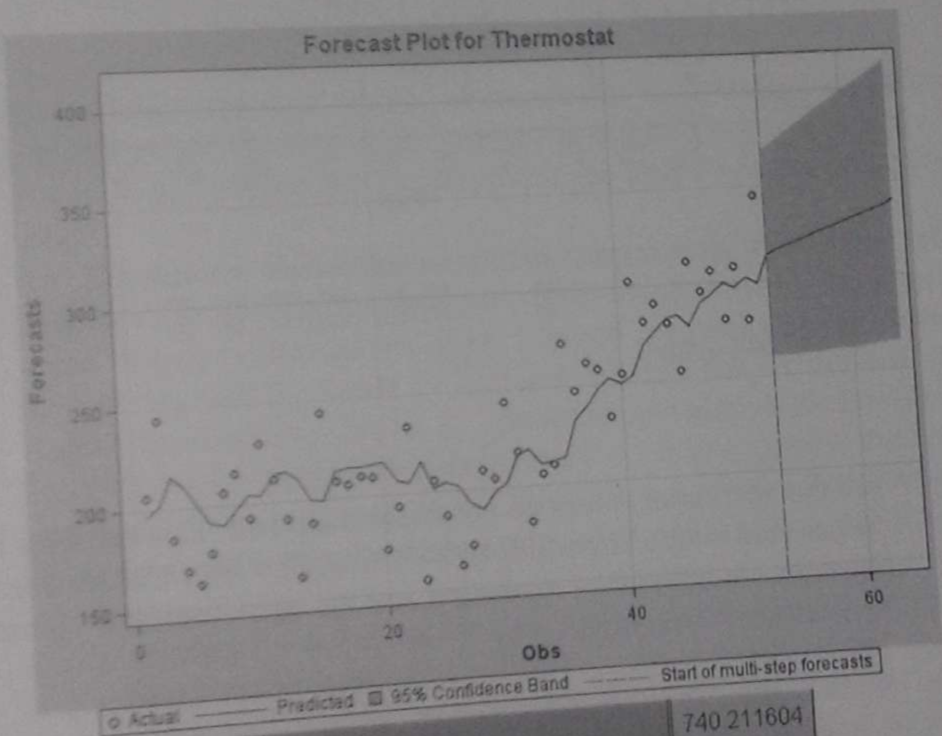
$$b_T = b_{T-1} + \alpha\gamma[y_T - (\ell_{T-1} + b_{t-1})]$$

In this form of the smoothing equations, we can see that both the estimate ℓ_{T-1} of the level and the estimate b_{T-1} of the growth rate are revised upward or downward depending on the sign of $[y_T - (\ell_{T-1} + b_{T-1})]$, which is the one-period-ahead forecast error.

Another type of exponential smoothing that has been applied to time series that exhibit a trend is Brown's double exponential smoothing. This smoothing method uses one smoothing constant to adjust both the level and the growth rate. We have chosen to examine the exponential smoothing methods that have corresponding state space models (see optional Section 8.6). The models can be used to justify the prediction intervals.

Software that provides simple exponential smoothing will most likely include Holt's trend corrected exponential smoothing. Since the software packages use different procedures for picking the initial values, finding the optimal smoothing constants, and computing prediction intervals, it is important to ask how these things are done. Figure 8.8 presents output from two such software packages. In the SAS Time Series Forecasting output of Figure 8.8(a) we see that $\gamma = .001$. This is the smallest value allowed in SAS Time Series Forecasting and indicates that the growth rate is not changing. It actually is quite common for a time series to have a level that changes without the growth rate changing. Thus, a time series can appear to have a changing trend when only the level is changing. In the MINITAB output of Figure 8.8(b) we can

FIGURE 8.8 Holt's trend corrected exponential smoothing for thermostat sales



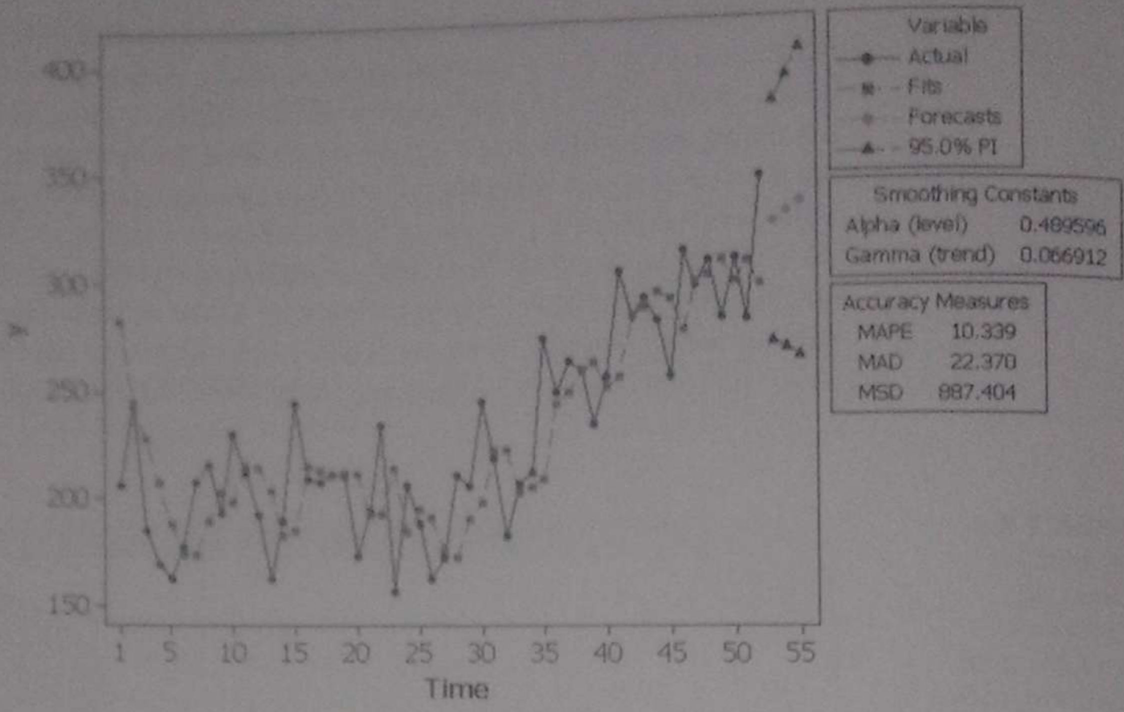
Mean Square Error	740.211604

Parameter	Estimate	Standard Error	t Value	Approx Pr > |t|
Level Weight	0.28080	0.06754	4.16	0.0001
Trend Weight	0.0010000	0.04299	0.02	0.9815

Obs	Forecasts	Standard Error	95% Confidence Limits	
53	313.8794	27.7456	259.4990	368.2599
54	316.2283	28.8209	259.7384	372.7141
55	318.5731	29.8594	260.0496	377.0965

(a) Time Series Forecasting in SAS

FIGURE 8.8
(Continued)



53	324.219	269.414	379.025
54	328.441	266.602	390.280
55	332.662	263.054	402.270

(b) MINITAB output

see that the initial values do not fit the data well. This causes a large α (=.4896) to be selected with a large SSE (= 887.404(52) = 46,145). As a result, the prediction intervals are wider than necessary, and the large value for y_{52} may have caused the forecasts to be too high.

8.4 HOLT–WINTERS METHODS

In this section we examine two Holt–Winters methods. Both methods are designed for time series that exhibit linear trend at least locally, if not over the range of the entire time series. The **additive Holt–Winters method** is used for time series with constant (additive) seasonal variation, whereas the **multiplicative Holt–Winters method** is used for time series with increasing (multiplicative) seasonal variation. These two types of seasonal variation are illustrated in Section 6.3. Unlike the approach in Chapter 6 for increasing seasonal variation, the multiplicative Holt–Winters method does not require a transformation of the time series but instead models the increasing seasonal variation directly. The multiplicative Holt–Winters method is the better known of the two methods and seems to be preferred if only one of the methods is employed. We look at the additive Holt–Winters method first because it is a linear method (all the components are added), and hence it is the simpler of the two methods. However, a reader can skip the additive Holt–Winters method and study the multiplicative Holt–Winters method first without any loss of information.

Additive Holt–Winters Method

If a time series has a linear trend with a fixed growth rate, β_1, and a fixed seasonal pattern, SN_t, with constant (additive) variation, then the time series may be described by the model

$$y_t = (\beta_0 + \beta_1 t) + SN_t + \varepsilon_t$$

In time series regression models, we use dummy variables to model SN_t (see Chapter 6). For this model, the level of the time series at time $T-1$ is $\beta_0 + \beta_1(T-1)$ and at time T is $\beta_0 + \beta_1 T$. Hence, the growth rate in the level from one time period to the next is β_1.

The additive Holt–Winters method is appropriate when a time series has a linear trend with an additive seasonal pattern for which the level, the growth rate, and the seasonal pattern *may be changing*. A model for these changing components of the time series can be found in optional Section 8.6. To implement the additive Holt–Winters method, we let ℓ_{T-1} denote the estimate of the **level** in time $T-1$, and b_{T-1} will denote the **growth rate** in time $T-1$. Then, suppose that we observe a new time series value y_T in time period T, and let sn_{T-L} denote the "most recent" estimate of the **seasonal factor** for the season corresponding to time period T. Here L denotes the number of seasons in a year ($L = 12$ for monthly data, and $L = 4$ for quarterly data), and thus $T-L$ denotes the time period occurring one year prior to time period T. Furthermore, the subscript $T-L$ of sn_{T-L} denotes the fact that the time series value in time period $T-L$ is the most recent time series value observed in the season being analyzed and thus is the most recent time series value used to help find sn_{T-L}. Then, the estimate of the level of the time series in time period T uses the smoothing constant α and is

$$\ell_T = \alpha(y_T - sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

where $(y_T - sn_{T-L})$ is the deseasonalized observation in time period T. The estimate of the growth rate in time period T uses the smoothing constant γ and is

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

The new estimate for the seasonal factor SN_T in time period T uses the smoothing constant δ and is

$$sn_T = \delta(y_T - \ell_T) + (1-\delta)sn_{T-L}$$

where $(y_T - \ell_T)$ is an estimate of the newly observed seasonal variation.

We summarize the additive Holt–Winters method in the following box.

ADDITIVE HOLT–WINTERS METHOD

1. Suppose that the time series $y_1, y_2, \ldots, y_n$ exhibits linear trend locally and has a seasonal pattern with constant (additive) seasonal variation and that the level, growth rate, and seasonal pattern may be changing. Then **the estimate ℓ_T for the level, the estimate b_T for the growth rate,** and **the estimate sn_T for the seasonal factor of the time series in time period T** are given by the smoothing equations

$$\ell_T = \alpha(y_T - \text{sn}_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

$$\text{sn}_T = \delta(y_T - \ell_T) + (1-\delta)\text{sn}_{T-L}$$

where α, γ, and δ are smoothing constants between 0 and 1, ℓ_{T-1} and b_{T-1} are estimates in time period $T-1$ for the level and growth rate, and sn_{T-L} is the estimate in time period $T-L$ for the seasonal factor.

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = \ell_T + \tau b_T + \text{sn}_{T+\tau-L} \qquad (\tau = 1, 2, \ldots)$$

where $\text{sn}_{T+\tau-L}$ is the "most recent" estimate of the seasonal factor for the season corresponding to time period $T + \tau$.

3. **A 95% prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]} s \sqrt{c_\tau}\right]$$

If $\tau = 1$ then $c_1 = 1$

If $2 \le \tau \le L$ then $c_\tau = \left[1 + \sum_{j=1}^{\tau-1} \alpha^2(1 + j\gamma)^2\right]$

If $L \le \tau$ then $c_\tau = 1 + \sum_{j=1}^{\tau-1} [\alpha(1 + j\gamma) + d_{j,L}(1-\alpha)\delta]^2$

where $d_{j,L} = 1$ if j is an integer multiple of L and 0 otherwise

The standard error s computed in time period T is

$$s = \sqrt{\frac{\text{SSE}}{T-3}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - \hat{y}_t(t-1)]^2}{T-3}} = \sqrt{\frac{\sum_{t=1}^{T}[y_t - (\ell_{t-1} + b_{t-1} + \text{sn}_{t-L})]^2}{T-3}}$$

The three smoothing equations of the additive Holt–Winters method can be put in the error correction form. Either form of the smoothing equations may be used to implement exponential smoothing (for example, when setting up a spreadsheet in Excel). Using the error correction form of the smoothing equations does not alter the choice of the smoothing parameters that minimize the SSE. Moreover, the formulas for the point forecasts and the 95% prediction intervals remain the same. It is, however, easier to relate the error correction form of the equations to the state space models of optional Section 8.6. The error correction form of the smoothing equations for the additive Holt–Winters method is given in the following box.

ERROR CORRECTION FORM

The error correction form for the smoothing equations in the **additive Holt–Winters method:**

$$\ell_T = \ell_{T-1} + b_{T-1} + \alpha[y_T - (\ell_{T-1} + b_{T-1} + sn_{T-L})]$$
$$b_T = b_{T-1} + \alpha\gamma[y_T - (\ell_{T-1} + b_{T-1} + sn_{T-L})]$$
$$sn_T = sn_{T-L} + (1-\alpha)\delta[y_T - (\ell_{T-1} + b_{T-1} + sn_{T-L})]$$

EXAMPLE 8.4 Consider the quarterly sales of the TRK-50 mountain bike presented in Exercise 6.3 of Chapter 6. Table 6.8 presents four years of quarterly sales of the TRK-50 mountain bike for the previous four years at a bicycle shop in Switzerland. The mountain bike sales are plotted in Figure 6.32. This plot suggests that the mountain bike sales display a linear demand and constant (additive) seasonal variation. Thus, we apply the additive Holt–Winters method to these data in order to find forecasts of future mountain bike sales.

We begin by finding estimates for the initial level, trend, and four seasonal factors. For seasonal data we need to use at least four or five years of data to find estimates for the seasonal factors. Hence, we first fit a least squares regression line to all four years of the available data rather than just half the data. As in Holt's trend corrected exponential smoothing, we let the *y*-intercept be ℓ_0 and the slope of the regression line be b_0. For example, the Excel regression output in Figure 8.9 gives the following least squares regression equation:

$$\hat{y}_t = 20.85 + .980882t$$

Thus, we choose $\ell_0 = 20.85$ and $b_0 = .9809$. These values have been copied to the cells that correspond to the level and growth rate at time 0 in the Excel spreadsheet of Figure 8.9.

The seasonal factors are found by the following three-step procedure:

1. We use the least squares regression equation to compute $\hat{y}_t$ for each time period that is used in finding the least squares regression equation. In our case, we compute $\hat{y}_t$ for the four years of data, $t = 1, 2, \ldots, 16$.

FIGURE 8.9 Excel spreadsheet of the additive Holt–Winters method for quarterly mountain bike sales, $\alpha = .20$, $\gamma = .10$, $\delta = .10$

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	n	alpha	gamma	delta		SSE	ssquare	s					
2	16	0.2	0.1	0.1		25.2166	1.9397	1.3927					
3													
4		Actual				Forecast		Squared					
5		Bike		Growth	Seasonal	Made Las	Forecast	Forecast				SUMMARY OUTPUT	
6	Time	Sales	Level	Rate	Factor	Period	Error	Error					
7	-3				-14.2162								
8	-2				6.5529							*Regression Statistics*	
9	-1				18.5721				Regression			Multiple R	0.320509
10	0		20.8500	0.9809	-10.9088				Estimates	Detrended	Average	R Square	0.102726
11	1	10	22.3079	1.0286	-14.0254	7.6147	2.3853	5.6896	21.8309	-11.8309	-14.2162	Adjusted R	0.038635
12	2	31	23.5586	1.0508	6.6418	29.8895	1.1105	1.2333	22.8118	8.1882	6.5529	Standard E	14.28614
13	3	43	24.5731	1.0472	18.5575	43.1815	-0.1815	0.0329	23.7926	19.2074	18.5721	Observatio	16
14	4	16	25.8780	1.0729	-10.8057	14.7115	1.2885	1.6603	24.7735	-8.7735	-10.9088		
15	5	11	26.5658	1.0344	-14.1794	12.9256	-1.9256	3.7079	25.7544	-14.7544	0.0000	ANOVA	
16	6	33	27.3518	1.0096	6.5424	34.2420	-1.2420	1.5427	26.7353	6.2647			*df*
17	7	45	27.9776	0.9712	18.4040	46.9190	-1.9190	3.6825	27.7162	17.2838		Regressior	1
18	8	17	28.7202	0.9483	-10.8972	18.1431	-1.1431	1.3067	28.6971	-11.6971		Residual	14
19	9	14	29.3707	0.9186	-14.2985	15.4892	-1.4892	2.2176	29.6779	-15.6779		Total	15
20	10	36	30.1230	0.9019	6.4759	36.8317	-0.8317	0.6918	30.6588	5.3412			
21	11	50	31.1391	0.9133	18.4497	49.4289	0.5711	0.3262	31.6397	18.3603			*Coefficient*
22	12	21	32.0214	0.9102	-10.9096	21.1553	-0.1553	0.0241	32.6206	-11.6206		Intercept	20.85
23	13	19	33.0050	0.9176	-14.2692	18.6331	0.3669	0.1346	33.6015	-14.6015		X Variable	0.980882
24	14	41	34.0429	0.9296	6.5240	40.3985	0.6015	0.3618	34.5824	6.4176			
25	15	55	35.2881	0.9612	18.5759	53.4222	1.5778	2.4894	35.5632	19.4368			
26	16	25	36.1813	0.9544	-10.9368	25.3396	-0.3396	0.1153	36.5441	-11.5441			

These values are shown in the column of "regression estimates." For example,

$$\hat{y}_1 = 20.85 + .980882(1) = 21.8309$$

and

$$\hat{y}_5 = 20.85 + .980882(5) = 25.7544$$

2. Next we detrend the data by computing $y_t - \hat{y}_t$ for each time period that is used to estimate the least squares regression line. For the mountain bike data, we compute $y_t - \hat{y}_t$ for the four years of data, $t = 1, 2, \ldots, 16$.

   These values are shown in the column of Figure 8.9 called "Detrended." For example,

$$y_1 - \hat{y}_1 = 10 - 21.8309 = -11.8309$$

and

$$y_5 - \hat{y}_5 = 11 - 25.7544 = -14.7544$$

3. Finally, the initial seasonal factor in each of the L seasons, $sn_{1-L}, sn_{2-L}, \ldots, sn_{-1}, sn_0$, is found by computing the average of the detrended values for the

corresponding season. For the mountain bike data we must find $L = 4$ seasonal factors, sn_{-3}, sn_{-2}, sn_{-1}, sn_0. These values are computed under "Average" in Figure 8.9 and have been copied to the cells for $t = -3, -2, -1$, and 0 under "Seasonal Factor."

For quarter 1, there are four first quarters of detrended data. Hence,

$$sn_{-3} = \frac{(y_1 - \hat{y}_1) + (y_5 - \hat{y}_5) + (y_9 - \hat{y}_9) + (y_{13} - \hat{y}_{13})}{4}$$

$$= \frac{-11.8309 + (-14.7544) + (-15.6779) + (-14.6015)}{4}$$

$$= -14.2162$$

Similarly, we find (by copying the Excel formula) that

$sn_{-2} = 6.5529$ is the seasonal factor for quarter 2

$sn_{-1} = 18.5721$ is the seasonal factor for quarter 3

$sn_0 = -10.9088$ is the seasonal factor for quarter 4

We want the average of the L seasonal factors to be 0. Notice that our four initial seasonal factors for the mountain bike sales do have an average of 0, and the sum is always 0 when using these three steps.

After finding the initial values for the level, trend, and four seasonal factors, we are ready to use the smoothing equations. In this example, we use the error correction form of the smoothing equations. Either form of the smoothing equations will produce the same numbers as in Figure 8.9. Starting with the initial values, we calculate a point forecast of y_1 from time origin 0 to be

$$\hat{y}_1(0) = \ell_0 + b_0 + sn_{1-4} = \ell_0 + b_0 + sn_{-3}$$
$$= 20.85 + .9089 + (-14.2162)$$
$$= 7.6147$$

This point forecast is shown in Figure 8.9 under "Forecast Made Last Period." Also shown in the spreadsheet is the actual mountain bike sales value $y_1 = 10$ and the forecast error, which is

$$y_1 - \hat{y}_1(0) = 10 - 7.6147 = 2.3853$$

The spreadsheet in Figure 8.9 is set up using $\alpha = .2$, $\gamma = .1$, and $\delta = .1$. Using $y_1 = 10$ and the error correction equation for ℓ_T, the estimate of the level of the time series in time period 1 is

$$\ell_1 = \ell_0 + b_0 + \alpha[y_1 - (\ell_0 + b_0 + sn_{1-4})]$$
$$= \ell_0 + b_0 + \alpha[y_1 - \hat{y}_1(0)]$$
$$= 20.85 + .9809 + .2(2.3853)$$
$$= 22.3079$$

Using the error correction equation for b_T, the estimate for the growth rate in time period 1 is

$$\begin{aligned} b_1 &= b_0 + \alpha\gamma[y_1 - (\ell_0 + b_0 + sn_{1-4})] \\ &= b_0 + \alpha\gamma[y_1 - \hat{y}_1(0)] \\ &= .9809 + (.2)(.1)(2.3853) \\ &= 1.0286 \end{aligned}$$

Using the error correction equation for sn_T, the estimate for the seasonal factor in time period 1 is

$$\begin{aligned} sn_1 &= sn_{1-4} + (1-\alpha)\delta[y_1 - (\ell_0 + b_0 + sn_{1-4})] \\ &= sn_{-3} + (.8)(.1)[y_1 - \hat{y}_1(0)] \\ &= -14.2162 + (.08)(2.3853) \\ &= -14.0254 \end{aligned}$$

It follows that a point forecast of y_2 in time period 1 is

$$\begin{aligned} \hat{y}_2(1) = \ell_1 + b_1 + sn_{2-4} &= \ell_1 + b_1 + sn_{-2} \\ &= 22.3079 + 1.0286 + 6.5529 \\ &= 29.8895 \end{aligned}$$

Since the actual mountain bike sales in time period 2 is $y_2 = 31$, the forecast error is

$$y_2 - \hat{y}_2(1) = 31 - 29.8895 = 1.1105$$

We continue this process for all 16 time periods. We also compute the squared forecast errors and the sum of the squared forecast errors (SSE). The results of this process are displayed in Figure 8.9, where we can see that SSE = 25.2166.

To find "good" values to use for α, γ, and δ, we select the values that minimize the sum of the squared forecast errors (SSE). Figure 8.10 shows the results of using Solver to find the minimum SSE. We see that the minimum SSE = 18.7975 is obtained using $\alpha = .561$, $\gamma = 0$, and $\delta = 0$, and that $s = 1.2025$. We also see that the final estimates for the level, growth rate, and seasonal factors are $\ell_{16} = 36.3426$, $b_{16} = .9809$, $sn_{13} = -14.2162$, $sn_{14} = 6.5529$, $sn_{15} = 18.5721$, and $sn_{16} = -10.9088$. Since the smoothing constants for the growth rate and the seasonal factors are both 0, these estimates have not changed from the initial estimates.

We now look at the process of computing point forecasts and 95% prediction intervals. We use the estimates from Figure 8.10 that produced the minimum SSE. The point forecasts of y_{17}, y_{18}, and y_{19} are

$$\begin{aligned} \hat{y}_{17}(16) &= \ell_{16} + b_{16} + sn_{17-4} \\ &= \ell_{16} + b_{16} + sn_{13} = 36.3426 + .9809 - 14.2162 = 23.1073 \\ \hat{y}_{18}(16) &= \ell_{16} + 2b_{16} + sn_{14} = 36.3426 + 2(.9809) + 6.5529 = 44.8573 \\ \hat{y}_{19}(16) &= \ell_{16} + 3b_{16} + sn_{15} = 36.3426 + 3(.9809) + 18.5721 = 57.8574 \end{aligned}$$

FIGURE 8.10 Excel spreadsheet giving the minimum SSE with the values of α, γ, δ, s, ℓ_{16}, b_{16}, and sn_{16}

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	n	alpha	gamma	delta		SSE	ssquare	s					
2	16	0.561	0	0		18.7975	1.4460	1.2025					
3													
4		Actual				Forecast		Squared					
5		Bike		Growth	Seasonal	Made Las	Forecast	Forecast					
6	Time	Sales	Level	Rate	Factor	Period	Error	Error				SUMMARY OUTPUT	
7	-3				-14.2162								
8	-2				6.5529							*Regression Statistics*	
9	-1				18.5721				Regression			Multiple R	0.320509
10	0		20.8500	0.9809	-10.9088				Estimates	Detrended	Average	R Square	0.102726
11	1	10	23.1682	0.9809	-14.2162	7.6147	2.3853	5.6896	21.8309	-11.8309	-14.2162	Adjusted R	0.038635
12	2	31	24.3161	0.9809	6.5529	30.7020	0.2980	0.0888	22.8118	8.1882	6.5529	Standard E	14.28614
13	3	43	24.8098	0.9809	18.5721	43.8691	-0.8691	0.7553	23.7926	19.2074	18.5721	Observatio	16
14	4	16	26.4176	0.9809	-10.9088	14.8818	1.1182	1.2503	24.7735	-8.7735	-10.9088		
15	5	11	26.1750	0.9809	-14.2162	13.1823	-2.1823	4.7622	25.7544	-14.7544	0.0000	ANOVA	
16	6	33	26.7585	0.9809	6.5529	33.7088	-0.7088	0.5024	26.7353	6.2647			*df*
17	7	45	27.0041	0.9809	18.5721	46.3114	-1.3114	1.7198	27.7162	17.2838		Regressior	1
18	8	17	27.9423	0.9809	-10.9088	17.0762	-0.0762	0.0058	28.6971	-11.6971		Residual	14
19	9	14	28.5266	0.9809	-14.2162	14.7070	-0.7070	0.4998	29.6779	-15.6779		Total	15
20	10	36	29.4737	0.9809	6.5529	36.0606	-0.0606	0.0037	30.6588	5.3412			
21	11	50	31.0003	0.9809	18.5721	49.0266	0.9734	0.9474	31.6397	18.3603			*Coefficients*
22	12	21	31.9406	0.9809	-10.9088	21.0723	-0.0723	0.0052	32.6206	-11.6206		Intercept	20.85
23	13	19	33.0867	0.9809	-14.2162	18.7053	0.2947	0.0868	33.6015	-14.6015		X Variable	0.980882
24	14	41	34.2803	0.9809	6.5529	40.6205	0.3795	0.1440	34.5824	6.4176			
25	15	55	35.9153	0.9809	18.5721	53.8333	1.1667	1.3612	35.5632	19.4368			
26	16	25	36.3426	0.9809	-10.9088	25.9874	-0.9874	0.9749	36.5441	-11.5441			

Then, a 95% prediction interval for y_{17} is

$$
\begin{aligned}
\left[\hat{y}_{17}(16) \pm z_{[.025]} s\sqrt{c_1}\right] &= \left[23.1073 \pm 1.96(1.2025)\sqrt{1}\right] \\
&= [23.1073 \pm 2.3569] \\
&= [20.7504,\ 25.4642]
\end{aligned}
$$

a prediction interval for y_{18} is

$$
\begin{aligned}
\left[\hat{y}_{18}(16) \pm z_{[.025]} s\sqrt{c_2}\right] &= \left[44.8573 \pm 1.96(1.2025)\sqrt{1+\alpha^2(1+\gamma)^2}\right] \\
&= \left[44.8573 \pm 1.96(1.2025)\sqrt{1+(.561)^2(1+0)^2}\right] \\
&= [44.8573 \pm 2.7025] \\
&= [42.1548,\ 47.5598]
\end{aligned}
$$

and a prediction interval for y_{19} is

$$
\begin{aligned}
\left[\hat{y}_{19}(16) \pm z_{[.025]} s\sqrt{c_3}\right] &= \left[57.8574 \pm 1.96(1.2025)\sqrt{1+\alpha^2(1+\gamma)^2+\alpha^2(1+2\gamma)^2}\right] \\
&= \left[57.8574 \pm 1.96(1.2025)\sqrt{1+(.561)^2(1+0)^2+(.561)^2(1+2(0))^2}\right] \\
&= [57.8574 \pm 3.0086] \\
&= [54.8488,\ 60.8660]
\end{aligned}
$$

As in the case of simple exponential smoothing and Holt's trend corrected smoothing, the point forecasts and prediction intervals can be revised by using the smoothing equations to revise the level, growth rate, and seasonal factor. Alternatively, with the power of today's computers, one can easily repeat the entire process with the new observation added.

Observe that the formulas for the 95% prediction intervals for the Holt's trend corrected exponential smoothing and for the additive Holt–Winters method are the same for the first year, except that the point forecast $\hat{y}_{T+\tau}(T)$ differs by the seasonal factor. For quarterly data, the 95% prediction intervals for both types of exponential smoothing in the first year are as follows:

Quarter 1: $\hat{y}_{T+1}(T) \pm z_{[.025]}s$

Quarter 2: $\hat{y}_{T+2}(T) \pm z_{[.025]}s\sqrt{1 + \alpha^2(1+\gamma)^2}$

Quarter 3: $\hat{y}_{T+3}(T) \pm z_{[.025]}s\sqrt{1 + \alpha^2(1+\gamma)^2 + \alpha^2(1+2\gamma)^2}$

Quarter 4: $\hat{y}_{T+4}(T) \pm z_{[.025]}s\sqrt{1 + \sum_{j=1}^{3}[\alpha^2(1+j\gamma)^2]}$

In the second year, the 95% prediction intervals for the two exponential smoothing methods differ starting with the term that corresponds to the first quarter of the second year under the square root sign. The term for the additive Holt–Winters method is

$$[\alpha(1+4\gamma) + (1-\alpha)\delta]^2$$

and the term for Holt's trend corrected exponential smoothing would be

$$\alpha^2(1+4\gamma)^2$$

Specifically, the intervals given by Holt–Winters method for the first and second quarters of the second year are

Quarter 1: $\hat{y}_{T+5}(T) \pm z_{[.025]}s\sqrt{1 + \sum_{j=1}^{3}[\alpha^2(1+j\gamma)^2] + [\alpha(1+4\gamma) + (1-\alpha)\delta]^2}$

Quarter 2: $\hat{y}_{T+6}(T) \pm z_{[.025]}s\sqrt{1 + \sum_{j=1}^{3}[\alpha^2(1+j\gamma)^2] + [\alpha(1+4\gamma) + (1-\alpha)\delta]^2 + \alpha^2(1+5\gamma)^2}$

For monthly data the difference would appear in the first month of the second year.

Multiplicative Holt–Winters Method

If a time series has a linear trend with a fixed growth rate, β_1, and a fixed seasonal pattern, SN_t, with increasing (multiplicative) variation, the time series may be described by the multiplicative model

$$y_t = (\beta_0 + \beta_1 t) \times SN_t \times IR_t$$

Here IR_t is an irregular component, as discussed in Chapter 1. In the classical multiplicative decomposition method, we estimated the fixed seasonal factors, SN_t, by using a procedure involving centered moving averages (see Chapter 7). For this model the level at time $T-1$ is $\beta_0 + \beta_1(T-1)$, and the level at time T is $\beta_0 + \beta_1 T$, showing that the growth rate for the level is β_1.

The **multiplicative Holt–Winters method** is appropriate when a time series has a linear trend with a multiplicative seasonal pattern for which the level, growth rate, and the seasonal pattern *may be changing rather than being fixed*. In optional Section 8.6, we discuss a model for this method. To implement the multiplicative Holt–Winters method, we let ℓ_{T-1} denote the estimate of the **level** in time $T-1$, and we let b_{T-1} denote the estimate of the **growth rate** in time $T-1$. Then, suppose that we observe a new time series value y_T in time period T, and let sn_{T-L} denote the "most recent" estimate of the **seasonal factor** for the season corresponding to time period T. Here L denotes the number of seasons in a year ($L = 12$ for monthly data, and $L = 4$ for quarterly data), and thus $T - L$ denotes the time period occurring one year prior to time period T. Furthermore, the subscript $T - L$ of sn_{T-L} denotes the fact that the time series value in time period $T - L$ was the most recent time series value observed in the season being analyzed and thus was the most recent time series value used to help find sn_{T-L}. Then, the estimate of the level of the time series in time period T uses the smoothing constant α and is

$$\ell_T = \alpha(y_T/sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$

where (y_T/sn_{T-L}) is the deseasonalized observation in time period T. The estimate of the growth rate in time period T uses the smoothing constant γ and is

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$

The new estimate for the seasonal factor SN_T in time period T uses the smoothing constant δ and is

$$sn_T = \delta(y_T/\ell_T) + (1-\delta)sn_{T-L}$$

where (y_T/ℓ_T) is an estimate of the newly observed seasonal variation.

We summarize the multiplicative Holt–Winters method in the following box.

MULTIPLICATIVE HOLT–WINTERS METHOD

1. Suppose that the time series $y_1, y_2, \ldots, y_n$ exhibits linear trend locally and has a seasonal pattern with increasing (multiplicative) seasonal variation and that the level, growth rate, and seasonal pattern may be changing. Then **the estimate ℓ_T for the level, the estimate b_T for the growth rate, and the estimate sn_T for the seasonal factor of the time series in time period T** are given by the smoothing equations

$$\ell_T = \alpha(y_T/sn_{T-L}) + (1-\alpha)(\ell_{T-1} + b_{T-1})$$
$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)b_{T-1}$$
$$sn_T = \delta(y_T/\ell_T) + (1-\delta)sn_{T-L}$$

where α, γ, and δ are smoothing constants between 0 and 1, ℓ_{T-1} and b_{T-1} are estimates in time period $T-1$ for the level and growth rate, and sn_{T-L} is the estimate in time period $T-L$ for the seasonal factor.

2. **A point forecast made in time period T for $y_{T+\tau}$ is**

$$\hat{y}_{T+\tau}(T) = (\ell_T + \tau b_T)\text{sn}_{T+\tau-L} \qquad (\tau = 1, 2, \ldots)$$

where $\text{sn}_{T+\tau-L}$ is the "most recent" estimate of the seasonal factor for the season corresponding to time period $T + \tau$.

3. **An approximate 95% prediction interval computed in time period T for $y_{T+\tau}$ is**

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]} s_r \left(\sqrt{c_\tau}\right)(\text{sn}_{T+\tau-L})\right]$$

$$\text{if} \quad \tau = 1 \quad \text{then} \quad c_1 = (\ell_T + b_T)^2$$

$$\text{if} \quad \tau = 2 \quad \text{then} \quad c_2 = \alpha^2(1+\gamma)^2(\ell_T + b_T)^2 + (\ell_T + 2b_T)^2$$

$$\text{if} \quad \tau = 3 \quad \text{then} \quad c_3 = \alpha^2(1+2\gamma)^2(\ell_T + b_T)^2 + \alpha^2(1+\gamma)^2(\ell_T + 2b_T)^2 + (\ell_T + 3b_T)^2$$

$$\text{if} \quad 2 \le \tau \le L \quad \text{then}$$

$$\begin{aligned} c_\tau &= \sum_{j=1}^{\tau-1} \alpha^2(1 + [\tau - j]\gamma)^2(\ell_T + jb_T)^2 + (\ell_T + \tau b_T)^2 \\ &= \alpha^2(1 + [\tau - 1]\gamma)^2(\ell_T + b_T)^2 + \cdots \\ &\quad + \alpha^2(1+\gamma)^2(\ell_T + [\tau - 1]b_T)^2 + (\ell_T + \tau b_T)^2 \end{aligned}$$

The relative standard error s_r computed in time period T is

$$s_r = \sqrt{\frac{\sum_{t=1}^{T}\left[\frac{y_t - \hat{y}_t(t-1)}{\hat{y}_t(t-1)}\right]^2}{T-3}} = \sqrt{\frac{\sum_{t=1}^{T}\left[\frac{y_t - (\ell_{t-1} + b_{t-1})\text{sn}_{t-L}}{(\ell_{t-1} + b_{t-1})\text{sn}_{t-L}}\right]^2}{T-3}}$$

(For a better approximation and an exact formula, see Hyndman et al., 2001.)

The three smoothing equations of the multiplicative Holt–Winters method can be put in the error correction form. Either form of the smoothing equations may be used to implement exponential smoothing (for example, when setting up a spreadsheet in Excel). Using the error correction form of the smoothing equations does not alter the choice of the smoothing parameters that minimize the SSE. Moreover, the formulas for the point forecasts and the 95% prediction intervals remain the same. It is, however, easier to relate the error correction form of the equations to the state space models of optional Section 8.6. The error correction form of the smoothing equations for the multiplicative Holt–Winters method is given in the following box.

ERROR CORRECTION FORM

The error correction form for the smoothing equations in the **multiplicative Holt–Winters method:**

$$\ell_T = \ell_{T-1} + b_{T-1} + \alpha \frac{[y_T - (\ell_{T-1} + b_{T-1})sn_{T-L}]}{sn_{T-L}}$$

$$b_T = b_{T-1} + \alpha\gamma \frac{[y_T - (\ell_{T-1} + b_{T-1})sn_{T-L}]}{sn_{T-L}}$$

$$sn_T = sn_{T-L} + (1-\alpha)\delta \frac{[y_T - (\ell_{T-1} + b_{T-1})sn_{T-L}]}{\ell_T}$$

EXAMPLE 8.5 The quarterly sales of Tiger Sports Drink for the last eight years are given in Table 8.2 and a plot of the sales is shown in Figure 8.11. The plot indicates that there is a linear increase in sales over the eight-year period and that the seasonal pattern is increasing as the level of the time series increases. This pattern suggests that multiplicative Holt–Winters might be employed to forecast future sales.

We will use the smoothing equations to construct the spreadsheet in Figure 8.12. However, we must first find initial values for the level ℓ_0, the growth rate b_0, and the seasonal factors, sn_{-3}, sn_{-2}, sn_{-1}, and sn_0. We need data for at least four or five years to find

TABLE 8.2 Quarterly Sales of Tiger Sports Drink (1000s of Cases)

	Year							
Quarter	1	2	3	4	5	6	7	8
1	72	77	81	87	94	102	106	115
2	116	123	131	140	147	162	170	177
3	136	146	158	167	177	191	200	218
4	96	101	109	120	128	134	142	149

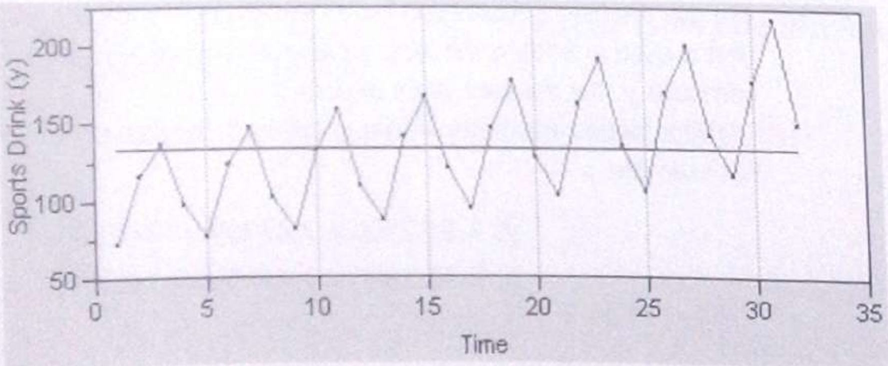


FIGURE 8.11 JMP IN plot of quarterly Tiger Sports Drink sales (1000s of cases)

FIGURE 8.12
Excel spreadsheet of the multiplicative Holt-Winters method for quarterly sales of Tiger Sports Drink, $\alpha = .2$, $\gamma = .1$, $\delta = .1$

	A	B	C	D	E	F	G	H	I
1	n	alpha	gamma	delta	SSE	ssquare	s	SSRE	sr
2	32	0.2	0.1	0.1	177.3233	6.1146	2.4728	0.0119	0.0202
3									
4						Forecast		Squared	Squared
5		Actual		Growth	Seasonal	Made Last	Forecast	Forecast	Relative
6	Time	Demand	Level	Rate	Factor	Period	Error	Error	Error
7	-3				0.7062				
8	-2				1.1114				
9	-1				1.2937				
10	0		95.2500	2.4706	0.8886				
11	1	72	98.5673	2.5553	0.7086	69.0103	2.9897	8.9384	0.001877
12	2	116	101.7726	2.6203	1.1142	112.3876	3.6124	13.0492	0.001033
13	3	136	104.5393	2.6349	1.2944	135.0531	0.9469	0.8966	0.000049
14	4	96	107.3464	2.6521	0.8892	95.2350	0.7650	0.5852	0.000065
15	5	77	109.7310	2.6254	0.7079	77.9479	-0.9479	0.8985	0.000148
16	6	123	111.9629	2.5860	1.1127	125.1919	-2.1919	4.8045	0.000307
17	7	146	114.1975	2.5509	1.2928	148.2750	-2.2750	5.1757	0.000235
18	8	101	116.1165	2.4877	0.8872	103.8091	-2.8091	7.8913	0.000732
39	29	115	161.2804	2.2519	0.7047	113.1314	1.8686	3.4918	0.000273
40	30	177	162.8178	2.1804	1.1046	180.9529	-3.9529	15.6253	0.000477
41	31	218	165.7889	2.2595	1.2928	212.8988	5.1012	26.0220	0.000574
42	32	149	167.8900	2.2437	0.8905	149.7057	-0.7057	0.4981	0.000022

values for the initial seasonal factors. Hence, in this example we use half of the data, 16 values, to find the initial values. As in Holt's trend corrected exponential smoothing, we use regression to fit a trend line to the first 16 sales values. In Figure 8.13, a regression output in Excel gives the following least squares regression equation:

$$\hat{y}_t = 95.2500 + 2.4706$$

Thus, for the initial values of the level and growth rate, we use $\ell_0 = 95.2500$ and $b_0 = 2.4706$. These values have been copied into the cells that correspond to the level and growth rate at time 0 in Figure 8.12.

The initial seasonal factors are found by the following four-step process:

1. We use the least squares regression equation to compute $\hat{y}_t$ for each time period that is used in finding the least squares regression equation. In our case, we compute $\hat{y}_t$ for the four years of data, $t = 1, 2, \ldots, 16$.

   These values are shown in the column of "regression estimates" in Figure 8.13. For example,

$$\hat{y}_1 = 95.2500 + 2.4706(1) = 97.7206$$
$$\hat{y}_2 = 95.2500 + 2.4706(2) = 100.1912$$

and

$$\hat{y}_5 = 95.2500 + 2.4706(5) = 107.6029$$

FIGURE 8.13 Using Excel to find initial estimates of the level ℓ_0, the growth rate b_0, and the seasonal factors $sn_{-3}, sn_{-2}, sn_{-1}, sn_0$

	A	B	C	D	E	F	G	H	I
1	Actual		Regression						
2	Demand	Quarter	Estimates	Detrended	Quarter 1	Quarter 2	Quarter 3	Quarter 4	Index
3	72	1	97.7206	0.7368	0.7368	1.1578	1.3247	0.9131	0.7062
4	116	2	100.1912	1.1578	0.7156	1.1174	1.2973	0.8781	1.1114
5	136	3	102.6818	1.3247	0.6894	1.0921	1.2906	0.8727	1.2937
6	96	4	105.1324	0.9131	0.6831	1.0783	1.2622	0.8903	0.8886
7	77	1	107.6029	0.7156					
8	123	2	110.0735	1.1174					
9	146	3	112.5441	1.2973		Averages by Quarter			Sum
10	101	4	115.0147	0.8781	0.7062	1.1114	1.2937	0.8886	3.9999
11	81	1	117.4853	0.6894		*df*	*SS*	*MS*	*F*
12	131	2	119.9559	1.0921	Regressio	1	2075.294	2075.294	2.727649
13	158	3	122.4265	1.2906	Residual	14	10651.71	760.8361	
14	109	4	124.8971	0.8727	Total	15	12727		
15	87	1	127.3676	0.6831					
16	140	2	129.8382	1.0783		*Coefficients*	*andard Err*	*t Stat*	*P-value*
17	167	3	132.3088	1.2622	Intercept	95.25	14.46478	6.584959	1.22E-05
18	120	4	134.7794	0.8903	X Variable	2.470588	1.495912	1.65156	0.120868

2. Next we detrend the data by computing $S_t = y_t/\hat{y}_t$ for each time period that is used to estimate the least squares regression line. For the sports drink data, we compute $S_t = y_t/\hat{y}_t$ for the four years of data, $t = 1, 2, \ldots, 16$.

These values are shown in the column called "Detrended" in Figure 8.13. For example,

$$S_1 = y_1/\hat{y}_1 = 72/97.7206 = .7368$$
$$S_2 = y_2/\hat{y}_2 = 116/100.1912 = 1.1578$$

and

$$S_5 = y_5/\hat{y}_5 = 77/107.6029 = .7156$$

3. Then, the average seasonal values are computed for each of the L seasons. These L averages, $\bar{S}_{[1]}, \bar{S}_{[2]}, \ldots, \bar{S}_{[L]}$, are found by computing the average of the detrended values for the corresponding season. For the sports drink data we must find $L = 4$ seasonal factors, $\bar{S}_{[1]}, \bar{S}_{[2]}, \bar{S}_{[3]}, \bar{S}_{[4]}$. For quarter 1, there are four first quarters of detrended data. Hence,

$$\bar{S}_{[1]} = \frac{(y_1/\hat{y}_1) + (y_5/\hat{y}_5) + (y_9/\hat{y}_9) + (y_{13}/\hat{y}_{13})}{4}$$
$$= \frac{S_1 + S_5 + S_9 + S_{13}}{4}$$
$$= \frac{.7368 + .7156 + .6894 + .6831}{4}$$
$$= .7062$$

Similarly, we find (by copying the Excel formula) that

$$\bar{S}_{[2]} = 1.1114 \text{ is the seasonal average for quarter 2}$$

$$\bar{S}_{[3]} = 1.2937 \text{ is the seasonal average for quarter 3}$$

$$\bar{S}_{[4]} = .8886 \text{ is the seasonal average for quarter 4}$$

4. Finally, we want the average of the seasonal factors to be 1. We do this by multiplying the average seasonal values by the correction factor

$$CF = \frac{L}{\sum_{i=1}^{L} \bar{S}_{[i]}}$$

Note that if $\Sigma_{i=1}^{L}\bar{S}_{[i]} = L$, there is no correction to be made because CF = 1. In order to use the smoothing equations in the spreadsheet in Figure 8.12, the initial seasonal factors have time subscripts for which the first season is in time period $1 - L$, the second season is in time period $2 - L$, and the last season is in time period $L - L = 0$. Thus the initial seasonal factors are

$$sn_{i-L} = \bar{S}_{[i]}(CF) \qquad (i = 1, 2, \ldots, L)$$

For the quarterly sports drinks data, we find four initial seasonal factors. In this case, CF = 4/3.9999 = 1.0000 and

$$sn_{-3} = sn_{1-4} = \bar{S}_{[1]}(CF) = .7062(1) = .7062$$
$$sn_{-2} = sn_{2-4} = \bar{S}_{[2]}(CF) = 1.1114(1) = 1.1114$$
$$sn_{-1} = sn_{3-4} = \bar{S}_{[3]}(CF) = 1.2937(1) = 1.2937$$
$$sn_{0} = sn_{4-4} = \bar{S}_{[4]}(CF) = .8886(1) = .8886$$

These values are listed under "Index" in Figure 8.13 and copied to the cells for time periods −3, −2, −1, and 0 under "Seasonal Factor" in Figure 8.12.

After finding the initial values for the level, trend, and four seasonal factors, we are ready to use the smoothing equations. Starting with the initial values, we calculate a point forecast of y_1 from time origin 0 to be

$$\begin{aligned}\hat{y}_1(0) &= (\ell_0 + b_0)sn_{1-4} = (\ell_0 + b_0)sn_{-3} \\ &= (95.2500 + 2.4706)(.7062) \\ &= 69.0103\end{aligned}$$

This point forecast is shown in Figure 8.12 under "Forecast Made Last Period." Also shown in the spreadsheet is the actual sales value $y_1 = 72$ and the forecast error, which is

$$y_1 - \hat{y}_1(0) = 72 - 69.0103 = 2.9897$$

The spreadsheet in Figure 8.12 is set up using $\alpha = .2$, $\gamma = .1$, and $\delta = .1$. In this example, we use the original form of the smoothing equations. Either the original or the error correction form of the smoothing equations will produce the same numbers as in Figure 8.12. Using $y_1 = 72$ and the smoothing equation for ℓ_T, the estimate of the level of the time series in time period 1 is

$$\begin{aligned}\ell_1 &= \alpha(y_1/sn_{1-4}) + (1-\alpha)(\ell_0 + b_0)\\ &= \alpha(y_1/sn_{-3}) + (1-\alpha)(\ell_0 + b_0)\\ &= .2(72/.7062) + .8(95.2500 + 2.4706)\\ &= 98.5673\end{aligned}$$

Using the smoothing equation for b_T, the estimate for the growth rate in time period 1 is

$$\begin{aligned}b_1 &= \gamma(\ell_1 - \ell_0) + (1-\gamma)b_0\\ &= .1(98.5673 - 95.2500) + .9(2.4706)\\ &= 2.5553\end{aligned}$$

Using the smoothing equation for sn_T, the estimate for the seasonal factor in time period 1 is

$$\begin{aligned}sn_1 &= \delta(y_1/\ell_1) + (1-\delta)sn_{1-4}\\ &= \delta(y_1/\ell_1) + (1-\delta)sn_{-3}\\ &= .1(72/98.5673) + .9(.7062)\\ &= .7086\end{aligned}$$

It follows that a point forecast of y_2 in time period 1 is

$$\begin{aligned}\hat{y}_2(1) &= (\ell_1 + b_1)sn_{2-4} = (\ell_1 + b_1)sn_{-2}\\ &= (98.5673 + 2.5553)(1.1114)\\ &= 112.3876\end{aligned}$$

Since the actual sales value in time period 2 is $y_2 = 116$, the forecast error is

$$y_2 - \hat{y}_2(1) = 116 - 112.3876 = 3.6124$$

Using the estimates that we just obtained, we can now compute the updated estimates ℓ_2, b_2, and sn_2 as follows:

$$\begin{aligned}\ell_2 &= \alpha(y_2/sn_{2-4}) + (1-\alpha)(\ell_1 + b_1)\\ &= \alpha(y_2/sn_{-2}) + (1-\alpha)(\ell_1 + b_1)\\ &= .2(116/1.1114) + .8(98.5673 + 2.5553)\\ &= 101.7726\end{aligned}$$

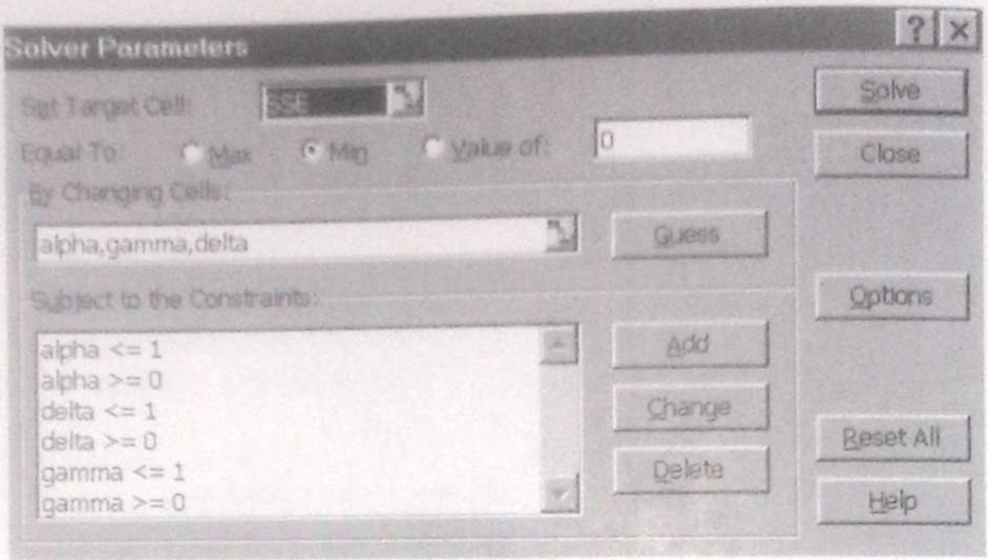
$$\begin{aligned}b_2 &= \gamma(\ell_2 - \ell_1) + (1-\gamma)b_1\\ &= .1(101.7726 - 98.5673) + .9(2.5553)\\ &= 2.6203\end{aligned}$$

$$\begin{aligned}sn_2 &= \delta(y_2/\ell_2) + (1-\delta)sn_{2-4}\\ &= \delta(y_2/\ell_2) + (1-\delta)sn_{-2}\\ &= .1(116/101.7726) + .9(1.1114)\\ &= 1.1142\end{aligned}$$

We continue this process for all 32 time periods. Note that the formulas for time period 1 in the spreadsheet can be copied for the remaining 31 time periods. We also compute the squared forecast errors and the sum of the squared forecast errors (SSE). The results of this process are displayed in Figure 8.12, where we can see that SSE = 177.3233.

To find "good" values to use for α, γ, and δ, we select the values that minimize the sum of the squared forecast errors (SSE). Figure 8.14 shows the results of using Solver to find the minimum SSE. We see that we obtain the minimum SSE = 168.4753 when $\alpha = .336$,

FIGURE 8.14
Finding α, γ, and δ values that minimize SSE using the multiplicative Holt–Winters method for sports drink sales



(a) Using Excel Solver for finding α (alpha), γ (gamma), and δ (delta) that minimize SSE

	A	B	C	D	E	F	G	H	I
1	n	alpha	gamma	delta	SSE	ssquare	s	SSRE	sr
2	32	0.336	0.046	0.134	168.4753	5.8095	2.4103	0.0108	0.0193
3									
4						Forecast		Squared	Squared
5		Actual		Growth	Seasonal	Made Last	Forecast	Forecast	Relative
6	Time	Demand	Level	Rate	Factor	Period	Error	Error	Error
7	-3				0.7062				
8	-2				1.1114				
9	-1				1.2937				
10	0		95.2500	2.4706	0.8886				
11	1	72	99.1415	2.5353	0.7089	69.0103	2.9897	8.9384	0.001877
12	2	116	102.5816	2.5765	1.1140	113.0036	2.9964	8.9786	0.000703
13	3	136	105.1470	2.5760	1.2937	136.0431	-0.0431	0.0019	0.000000
14	4	96	107.8277	2.5808	0.8888	95.7227	0.2773	0.0769	0.000008
⋮									
39	29	115	161.7496	2.3277	0.7044	113.1539	1.8461	3.4079	0.000266
40	30	177	162.7095	2.2655	1.1038	181.5084	-4.5084	20.3261	0.000617
41	31	218	166.2958	2.3256	1.2934	212.9210	5.0790	25.7960	0.000569
42	32	149	168.1213	2.3028	0.8908	150.3283	-1.3283	1.7643	0.000078

(b) Excel spreadsheet giving the minimum SSE with the values for α, γ, δ, ℓ_{32}, b_{32}, sn_{29}, sn_{30}, sn_{31}, and sn_{32}

$\gamma = .046$, and $\delta = .134$. We also see that the final estimates for the level, growth rate, and seasonal factors are $\ell_{32} = 168.1213$, $b_{32} = 2.3028$, $sn_{29} = .7044$, $sn_{30} = 1.1038$, $sn_{31} = 1.2934$, and $sn_{32} = .8908$.

We now look at the process for computing the point forecasts and 95% prediction intervals for the next year. Using the estimates from Figure 8.14(b) that minimize the SSE, the point forecasts of y_{33}, y_{34}, y_{35}, and y_{36} are

$$\begin{aligned}\hat{y}_{33}(32) &= (\ell_{32} + b_{32})sn_{33-4} \\ &= (\ell_{32} + b_{32})sn_{29} = (168.1213 + 2.3028)(.7044) = 120.0467 \\ \hat{y}_{34}(32) &= (\ell_{32} + 2b_{32})sn_{30} = [168.1213 + 2(2.3028)](1.1038) = 190.6560 \\ \hat{y}_{35}(32) &= (\ell_{32} + 3b_{32})sn_{31} = 226.3834 \\ \hat{y}_{36}(32) &= (\ell_{32} + 4b_{32})sn_{32} = 157.9678\end{aligned}$$

Before we compute the 95% prediction intervals, note that the formulas in the box for these 95% prediction intervals use the relative standard error s_r at time T rather than the standard error s at time T. The reason for this change is the result of using a multiplicative model where the trend is multiplied by both the seasonal factor and the irregular factor. To find s_r we find the sum of the squares of the relative errors $[y_t - \hat{y}_t(t-1)]/\hat{y}_t(t-1)$, $t = 1, 2, \ldots, T$, rather than the sum of the squares of the errors, $[y_t - \hat{y}_t(t-1)]$, as follows:

$$\begin{aligned}s_r &= \sqrt{\frac{\sum_{t=1}^{32}\left[\frac{y_t - \hat{y}_t(t-1)}{\hat{y}_t(t-1)}\right]^2}{32-3}} \\ &= \sqrt{\frac{\left[\frac{72 - 69.0103}{69.0103}\right]^2 + \left[\frac{116 - 113.0036}{113.0036}\right]^2 + \cdots + \left[\frac{149 - 150.3283}{150.3283}\right]^2}{29}} \\ &= \sqrt{\frac{SSRE}{29}} = \sqrt{\frac{.0108}{29}} = .0193 \qquad \text{(see Figure 8.14b)}\end{aligned}$$

Then a 95% prediction interval for y_{33} is

$$\begin{aligned}\left[\hat{y}_{33}(32) \pm z_{[.025]}s_r\left(\sqrt{c_1}\right)(sn_{33-4})\right] &= \left[120.0467 \pm 1.96(.0193)\left(\sqrt{(\ell_{32} + b_{32})^2}\right)(.7044)\right] \\ &= [120.0467 \pm 1.96(.0193)(168.1213 + 2.3028)(.7044)] \\ &= [120.0467 \pm 4.5411] \\ &= [115.5056, 124.5858]\end{aligned}$$

To compute the 95% prediction interval for y_{34} we first compute

$$\begin{aligned}c_2 &= \alpha^2(1+\gamma)^2(\ell_{32} + b_{32})^2 + (\ell_{32} + 2b_{32})^2 \\ &= (.336)^2(1 + .046)^2(168.1213 + 2.3028)^2 + (168.1213 + 2(2.3028))^2 \\ &= 33{,}422.1814\end{aligned}$$

Then,

$$\begin{aligned}&\left[\hat{y}_{34}(32) \pm z_{[.025]}s_r\left(\sqrt{c_2}\right)(sn_{34-4})\right]\\&= \left[190.6560 \pm 1.96(.0193)\left(\sqrt{33{,}422.1814}\right)(1.1038)\right]\\&= [190.6560 \pm 7.6335]\\&= [183.0225, 198.2895]\end{aligned}$$

Similarly, we find the 95% prediction interval for y_{35} is

$$\begin{aligned}&\left[\hat{y}_{35}(32) \pm z_{[.025]}s_r\left(\sqrt{c_3}\right)(sn_{35-4})\right]\\&= \left[226.3834 \pm 1.96(.0193)\left(\sqrt{38{,}230.6847}\right)(1.2934)\right]\\&= [226.3834 \pm 9.5665]\\&= [216.8169, 235.9499]\end{aligned}$$

and the 95% prediction interval for y_{36} is

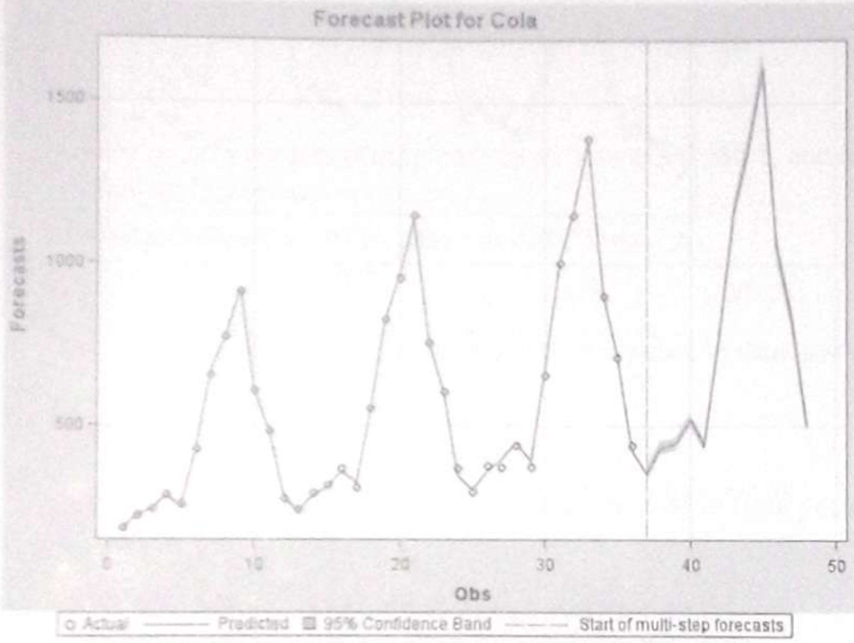
$$\left[\hat{y}_{36}(32) \pm z_{[.025]}s_r\left(\sqrt{c_4}\right)(sn_{36-4})\right] = [150.9402, 164.9954]$$

If new quarterly values for the sales of Tiger Sports Drink become available, we can revise our forecasts and 95% prediction intervals by using the smoothing equations to update our estimates for the level, growth rate, and seasonal factor without reestimating the smoothing parameters. Historically, the ability to make such revisions was an advantage of exponential smoothing methods because only the most recent estimates are needed for the revision. However, with the speed and capacity of today's computers, the values of new observations can be added to the data and new estimates for the smoothing parameters can be found even when we need to forecast thousands of items.

Both the additive and multiplicative versions of the Holt–Winters methods are normally available in exponential smoothing software. If only one version is available, it will most likely be the multiplicative version. As with the other exponential smoothing methods, one should ask how such things as initial values and prediction intervals are found. Figure 8.15 presents two examples of computer output from applying the multiplicative Holt–Winters method to the monthly sales of Tasty Cola (see Table 7.1 and Figure 7.2). Since the graph in Figure 7.2 shows that the sales of Tasty Cola exhibit a linear trend with increasing seasonal variation, the multiplicative Holt–Winters method is an appropriate method for this time series. Figure 8.15(a) presents the output from the default procedure in Time Series Forecasting in SAS, which uses an automatic optimization to find the smoothing constants. Figure 8.15(b) presents a MINITAB output where the default values for the smoothing constants ($\alpha = \gamma = \delta = .2$) were used. Holt–Winters methods in MINITAB do not have the option of an automatic search for optimal smoothing constants.

Notice the dramatic difference in the values of the SSE. The mean square error in Figure 8.15(a) is 88.02, and the comparable MSD in Figure 8.15(b) is 6812.61; both of the values are equal to SSE/36. A large SSE is always reflected in the length of the prediction intervals, and hence the intervals are much wider in Figure 8.15(b) than in Figure 8.15(a). It is suggested in an exercise to search for better smoothing constants in MINITAB. However, by looking at the beginning of the time series, we can see that the difference between the predicted values and the actual values is much greater in Figure 8.15(b) than in Figure 8.15(a). Hence, it is evident that the initial estimates of

FIGURE 8.15 Multiplicative Holt–Winters method for monthly sales of Tasty Cola



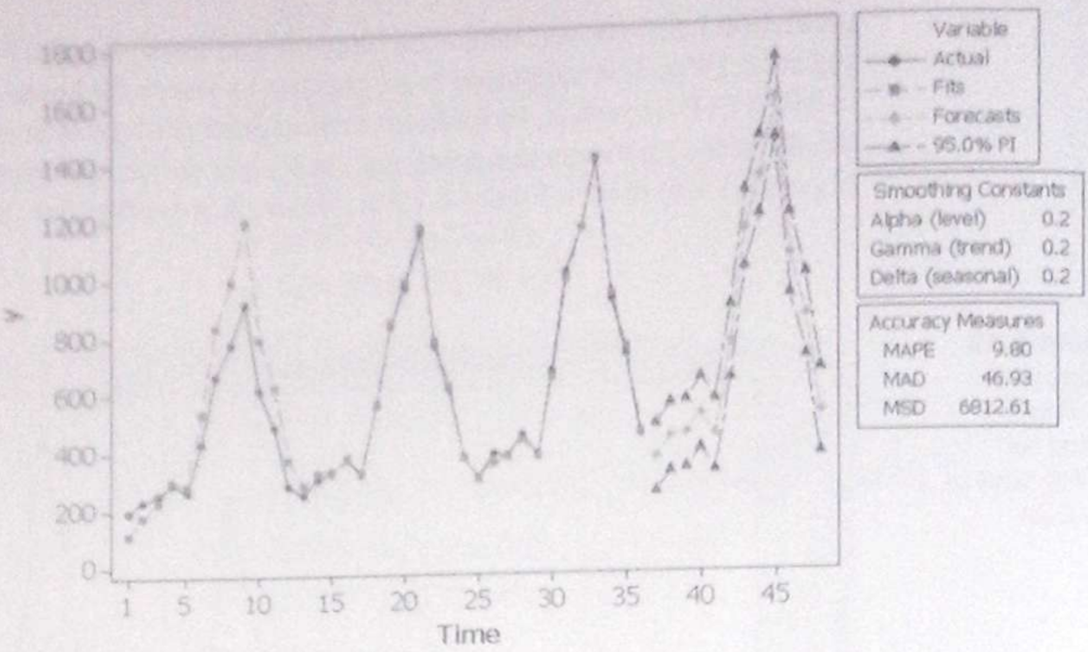
Mean Square Error	88.0248578

Parameter	Estimate	Standard Error	t Value	Approx Pr > |t|
Level Weight	0.11432	0.01249	9.15	<.0001
Trend Weight	0.24642	0.02346	10.50	<.0001
Seasonal Weight	0.56186	0.07396	7.60	<.0001

Obs	Forecasts	Standard Error	95% Confidence Limits	
37	352.0312	9.7993	332.8248	371.2375
38	434.5401	9.9460	415.0464	454.0339
39	445.0473	10.1128	425.2266	464.8679
40	521.2108	10.4983	500.6385	541.7831
41	444.6318	10.5304	423.9926	465.2709
42	775.4144	12.6263	750.6673	800.1615

(a) Time Series Forecasting in SAS

FIGURE 8.15
Continued



Period	Forecast	Lower	Upper
37	355.96	240.98	470.95
38	426.31	309.52	543.10
39	436.69	317.90	555.49
40	505.60	384.60	626.60
41	431.71	308.32	555.10

(b) MINITAB output

the level, growth rate, and seasonal factors were found by different procedures. For a short time series, the poor fit at the beginning of the time series will not be negligible and will result in wide prediction intervals.

8.5 DAMPED TREND AND OTHER EXPONENTIAL SMOOTHING METHODS

The methods presented so far include the most commonly used exponential methods: simple exponential smoothing, Holt's trend corrected exponential smoothing, the additive Holt–Winters method, and the multiplicative Holt–Winters method. Another common exponential smoothing method is Gardner and McKenzie's **damped trend exponential smoothing.** The damped trend method is appropriate for forecasting a time series which has a growth rate that will not be sustained into the future and whose effects should be dampened. Dampening the growth rate means to reduce it in size so that the rate of increase or decrease for the forecasts is slowing down. We first consider a method for damped trend when there is no seasonal pattern.

DAMPED TREND METHOD

1. Suppose that the time series $y_1, y_2, \ldots, y_n$ exhibits a linear trend for which the level and growth rate are changing somewhat with no seasonal pattern. Furthermore, suppose that we question whether the growth rate at the end of the time series will continue into the future. Then the estimate ℓ_T for the **level** and the estimate b_T for the **growth rate** are given by the smoothing equations

$$\ell_T = \alpha y_T + (1-\alpha)(\ell_{T-1} + \phi b_{T-1})$$
$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)\phi b_{T-1}$$

where α and γ are **smoothing constants** between 0 and 1, and ϕ is a **damping factor** between 0 and 1.

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = \ell_T + (\phi b_T + \phi^2 b_T + \cdots + \phi^\tau b_T)$$

3. If $\tau = 1$, then a **95% prediction interval computed in time period T for y_{T+1}** is

$$[\hat{y}_{T+1}(T) \pm z_{[.025]}s]$$

If $\tau = 2$, then a **95% prediction interval computed in time period T for y_{T+2}** is

$$\left[\hat{y}_{T+2} \pm z_{[.025]}s\sqrt{1 + \alpha^2(1+\phi\gamma)^2}\right]$$

If $\tau = 3$, then a **95% prediction interval computed in time period T for y_{T+3}** is

$$\left[\hat{y}_{T+3}(T) \pm z_{[.025]}s\sqrt{1 + \alpha^2(1+\varphi\gamma)^2 + \alpha^2(1+\phi\gamma+\phi^2\gamma)^2}\right]$$

If $\tau \geq 4$, then a **95% prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]}s\sqrt{1 + \sum_{j=1}^{\tau-1} \alpha^2(1+\phi_j\gamma)^2}\right]$$

where $\phi_j = \phi + \phi^2 + \cdots + \phi^j$.

We can see that the effect of the growth rate b_T in the point forecast is reduced further for each additional time period in the future, provided ϕ is less than 1. For example, if $\phi = .7$, then

$$\hat{y}_{T+1} = \ell_T + .7b_T$$
$$\hat{y}_{T+2} = \ell_T + .7b_T + (.7)^2 b_T = \ell_T + .7b_T + .49b_T$$
$$\hat{y}_{T+3} = \ell_T + .7b_T + (.7)^2 b_T + (.7)^3 b_T = \ell_T + .7b_T + .49b_T + .343b_T$$

As with the other exponential smoothing methods, the smoothing equations can be put in the error correction form.

ERROR CORRECTION FORM

The error correction form for the smoothing equations in **damped trend exponential smoothing:**

$$\ell_T = \ell_{T-1} + \phi b_{T-1} + \alpha[y_T - (\ell_{T-1} + \phi b_{T-1})]$$
$$b_T = \phi b_{T-1} + \alpha\gamma[y_T - (\ell_{T-1} + \phi b_{T-1})]$$

For seasonal data we can use damped trend with either the additive Holt–Winters method or the multiplicative Holt–Winters method.

ADDITIVE HOLT–WINTERS WITH DAMPED TREND

1. The estimate ℓ_T for the **level,** the estimate b_T for the **growth rate,** and the estimate sn_T for the **seasonal factor** of the time series in time period T are given by the smoothing equations

$$\ell_T = \alpha(y_T - \text{sn}_{T-L}) + (1-\alpha)(\ell_{T-1} + \phi b_{T-1})$$
$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma)\phi b_{T-1}$$
$$\text{sn}_T = \delta(y_T - \ell_T) + (1-\delta)\text{sn}_{T-L}$$

where α, γ, and δ are **smoothing constants** between 0 and 1, and ϕ is a **damping factor** between 0 and 1. The error correction form of the smoothing equations is

$$\ell_T = \ell_{T-1} + \phi b_{T-1} + \alpha[y_T - (\ell_{T-1} + \phi b_{T-1} + \text{sn}_{T-L})]$$
$$b_T = \phi b_{T-1} + \alpha\gamma[y_T - (\ell_{T-1} + \phi b_{T-1} + \text{sn}_{T-L})]$$
$$\text{sn}_T = \text{sn}_{T-L} + (1-\alpha)\delta[y_T - (\ell_{T-1} + \phi b_{T-1} + \text{sn}_{T-L})]$$

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = \ell_T + (\phi b_T + \phi^2 b_T + \cdots + \phi^\tau b_T) + \text{sn}_{T+\tau-L}$$

where $\text{sn}_{T+\tau-L}$ is the "most recent" estimate of the seasonal factor for the season corresponding to time period $T + \tau$.

3. A 95% **prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]} s \sqrt{c_\tau}\right]$$

$$\text{If} \quad \tau = 1 \quad \text{then} \quad c_1 = 1$$

$$\text{If} \quad \tau \geq 2 \quad \text{then} \quad c_\tau = 1 + \sum_{j=1}^{\tau-1} [\alpha(1 + \phi_j \gamma) + d_{j,L}(1 - \alpha)\delta]^2$$

where $d_{j,L} = 1$ if j is an integer multiple of L and 0 otherwise and $\phi_j = \phi + \phi^2 + \cdots + \phi^j$

MULTIPLICATIVE HOLT–WINTERS METHOD WITH DAMPED TREND

1. The estimate ℓ_T for the **level,** the estimate b_T for the **growth rate,** and the estimate sn_T for the **seasonal factor** of the time series in time period T are given by the smoothing equations

$$\ell_T = \alpha(y_T/\text{sn}_{T-L}) + (1 - \alpha)(\ell_{T-1} + \phi b_{T-1})$$
$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1 - \gamma)\phi b_{T-1}$$
$$\text{sn}_T = \delta(y_T/\ell_T) + (1 - \delta)\text{sn}_{T-L}$$

where α, γ, and δ are **smoothing constants** between 0 and 1, and ϕ is a **damping factor** between 0 and 1. The error correction form of the smoothing equations is

$$\ell_T = \ell_{T-1} + \phi b_{T-1} + \alpha \frac{[y_T - (\ell_{T-1} + \phi b_{T-1})\text{sn}_{T-L}]}{\text{sn}_{T-L}}$$

$$b_T = \phi b_{T-1} + \alpha\gamma \frac{[y_T - (\ell_{T-1} + \phi b_{T-1})\text{sn}_{T-L}]}{\text{sn}_{T-L}}$$

$$\text{sn}_T = \text{sn}_{T-L} + (1 - \alpha)\delta \frac{[y_T - (\ell_{T-1} + \phi b_{T-1})\text{sn}_{T-L}]}{\ell_T}$$

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = (\ell_T + \phi b_T + \phi^2 b_T + \cdots + \phi^\tau b_T)\text{sn}_{T+\tau-L}$$

where $\text{sn}_{T+\tau-L}$ is the "most recent" estimate of the seasonal factor for the season corresponding to time period $T + \tau$.

3. **An approximate 95% prediction interval computed in time period T for $y_{T+\tau}$** is

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]} s_r \left(\sqrt{c_\tau}\right)(\text{sn}_{T+\tau-L})\right]$$

If $\tau = 1$ then $c_1 = (\ell_T + \phi b_T)^2$

If $2 \le \tau \le L$ then

$$\begin{aligned} c_\tau &= \sum_{j=1}^{\tau-1} \alpha^2(1 + [\tau - j]\gamma)^2(\ell_T + \phi_j b_T)^2 + (\ell_T + \phi_\tau b_T)^2 \\ &= \alpha^2(1 + [\tau - 1]\gamma)^2(\ell_T + \phi b_T)^2 + \cdots + \\ &\quad \alpha^2(1 + \gamma)^2(\ell_T + \phi_{\tau-1} b_T)^2 + (\ell_t + \phi_\tau b_T)^2 \end{aligned}$$

where $\phi_j = \phi + \phi^2 + \cdots + \phi^j$

(For a better approximation and an exact formula, see Hyndman et al., 2001.)

When using damped trend methods, initial estimates can be obtained by utilizing the procedures presented for the Holt's trend corrected exponential smoothing, additive Holt–Winters method, or the multiplicative Holt–Winters method. The choice of smoothing constants and the damping factor is made by choosing the values that minimize the sum of the squared one-period-ahead forecast errors (SSE).

We have introduced methods for cases with no trend, linear trend, or damped trend. We have also provided methods to deal with no seasonal pattern, an additive seasonal pattern, and a multiplicative seasonal pattern. There is an exponential smoothing method that allows for exponential trend by multiplying the level by the growth rate instead of adding them. It is possible to have any combination of the four types of trend and three types of seasonality. In the next box, we show one more combination: no trend and a multiplicative seasonal pattern.

NO TREND MULTIPLICATIVE HOLT–WINTERS METHOD

1. The estimate ℓ_T for the **level,** the estimate b_T for the **growth rate,** and the estimate sn_T for the **seasonal factor** of the time series in time period T are given by the smoothing equations

$$\ell_T = \alpha(y_T/\text{sn}_{T-L}) + (1 - \alpha)(\ell_{T-1})$$
$$\text{sn}_T = \delta(y_T/\ell_T) + (1 - \delta)\text{sn}_{T-L}$$

where α and δ are **smoothing constants** between 0 and 1.

2. **A point forecast made in time period T for $y_{T+\tau}$** is

$$\hat{y}_{T+\tau}(T) = (\ell_T)\text{sn}_{T+\tau-L}$$

where $\text{sn}_{T+\tau-L}$ is the "most recent" estimate of the seasonal factor for the season corresponding to time period $T + \tau$.

3. **An approximate 95% prediction interval computed in time period T for $y_{T+\tau}$** when $1 \le \tau \le L$ is

$$\left[\hat{y}_{T+\tau}(T) \pm z_{[.025]}s_r\left(\sqrt{1 + (\tau - 1)\alpha^2}\right)(\ell_T \text{sn}_{T+\tau-L})\right]$$

(For a better approximation and an exact formula, see Hyndman et al., 2001.)

*8.6 MODELS FOR EXPONENTIAL SMOOTHING AND PREDICTION INTERVALS

Every exponential smoothing method has a corresponding statistical model. Statistical models are necessary for deriving formulas for prediction intervals. We will use **state space models** with a single source of error for the exponential smoothing models. The formulas for the prediction intervals in the preceding five sections were derived by using the models of this section. Before presenting the state space models, we need to introduce some new notation to distinguish the true values in the exponential smoothing models from the estimates that are found in the exponential smoothing methods. The components of exponential smoothing are called "**states**" in state space models. The notation for the components (states) is

Component (state)	Model	Estimate
Level in time period t	L_t	ℓ_t
Growth rate in time period t	B_t	b_t
Seasonal factor in time period t	SN_t	sn_t

The smoothing constants in the previous sections are estimates. To keep the notation relatively simple we change notation slightly and require that all the estimates in the previous sections for smoothing constants and the damping factor have hats (For example, $\hat{\alpha}$ is the estimate of the true value α). The new notation is

* This section is optional.

	Parameter	Estimate
Smoothing constant for the level	α	$\hat{\alpha}$
Smoothing constant for the growth rate	γ	$\hat{\gamma}$
Smoothing constant for the season	δ	$\hat{\delta}$
Damping factor	ϕ	$\hat{\phi}$

Each state space equation has an observation equation and one or more state equations. The **observation equation** is an equation for the value y_t, which can be observed. The **state equations** show how the unobserved components (states), which are the level, growth rate, and seasonal factor, change from one time period to the next. The models have a random source of error ε_t. The error term ε_t has a value from a normal distribution that has mean zero [that is, $E(\varepsilon_t) = 0$] and a standard deviation σ [that is, $\text{Var}(\varepsilon_t) = \sigma^2$] that is the same for each and every time period. Moreover, the error terms $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ in different time periods are assumed to be statistically independent of each other.

The **state space models** for the single source of error models that we have studied in the previous sections of this chapter are presented in Table 8.3.

In order to see the relationship between the state space models and the exponential smoothing methods, one should relate the models to the error correction form of the smoothing equations. For example, looking at the model for simple exponential smoothing, we see from the observation equation that $\varepsilon_t = y_t - L_{t-1}$ and hence, the state equation is $L_t = L_{t-1} + \alpha(y_t - L_{t-1})$. Since ℓ_t is an estimate for the level L_t and $\hat{\alpha}$ is an estimate of α, we see that the smoothing equation, $\ell_t = \ell_{t-1} + \hat{\alpha}(y_t - \ell_{t-1})$, follows from the model.

TABLE 8.3 State Space Models for the Exponential Smoothing Methods

	Model	
Method	**Observation Equation**	**State Equations**
Simple exponential smoothing	$y_t = L_{t-1} + \varepsilon_t$	$L_t = L_{t-1} + \alpha\varepsilon_t$
Holt's trend corrected exponential smoothing	$y_t = L_{t-1} + B_{t-1} + \varepsilon_t$	$L_t = L_{t-1} + B_{t-1} + \alpha\varepsilon_t$ $B_t = B_{t-1} + \alpha\gamma\varepsilon_t$
Additive Holt–Winters method	$y_t = L_{t-1} + B_{t-1} + SN_{t-L} + \varepsilon_t$	$L_t = L_{t-1} + B_{t-1} + \alpha\varepsilon_t$ $B_t = B_{t-1} + \alpha\gamma\varepsilon_t$ $SN_t = SN_{t-L} + (1-\alpha)\delta\varepsilon_t$
Multiplicative Holt–Winters method	$y_t = (L_{t-1} + B_{t-1})\, SN_{t-L}\,(1 + \varepsilon_t)$	$L_t = L_{t-1} + B_{t-1} + \alpha\,(L_{t-1} + B_{t-1})\varepsilon_t$ $B_t = B_{t-1} + \alpha\gamma(L_{t-1} + B_{t-1})\varepsilon_t$ $SN_t = SN_{t-L} + (1-\alpha)\delta(SN_{t-L})\varepsilon_t$
Damped trend method	$y_t = L_{t-1} + \phi B_{t-1} + \varepsilon_t$	$L_t = L_{t-1} + \phi B_{t-1} + \alpha\varepsilon_t$ $B_t = \phi B_{t-1} + \alpha\gamma\varepsilon_t$
Additive Holt–Winters method with damped trend	$y_t = L_{t-1} + \phi B_{t-1} + SN_{t-L} + \varepsilon_t$	$L_t = L_{t-1} + \phi B_{t-1} + \alpha\varepsilon$ $B_t = \phi B_{t-1} + \alpha\gamma\varepsilon_t$ $SN_t = SN_{t-L} + (1-\alpha)\delta\varepsilon_t$
Multiplicative Holt–Winters method with damped trend	$y_t = (L_{t-1} + \phi B_{t-1})SN_{t-L}\,(1 + \varepsilon_t)$	$L_t = L_{t-1} + \phi B_{t-1} + \alpha(L_{t-1} + \phi B_{t-1})\varepsilon_t$ $B_t = \phi B_{t-1} + \alpha\gamma\,(L_t + \phi B_{t-1})\varepsilon_t$ $SN_t = SN_{t-L} + (1-\alpha)\delta SN_{t-L}\varepsilon_t$
No trend multiplicative Holt–Winters method	$y_t = L_{t-1}SN_{t-L}(1 + \varepsilon_t)$	$L_t = L_{t-1} + \alpha L_{t-1}\varepsilon_t$ $SN_t = SN_{t-L} + (1-\alpha)\delta SN_{t-L}\varepsilon_t$

The relationship between the state space model and the exponential smoothing method is not so obvious for the multiplicative Holt–Winters method, where the trend, $TR_t = L_{t-1} + B_{t-1}$, the seasonal factor, SN_{t-L}, and the irregular factor, $IR_t = 1 + \varepsilon_t$, are multiplied together. However, the procedure to see the relationship is the same. We can rewrite the observation equation as

$$y_t = (L_{t-1} + B_{t-1})SN_{t-L}(1 + \varepsilon_t) = (L_{t-1} + B_{t-1})SN_{t-L} + (L_{t-1} + B_{t-1})SN_{t-L}\varepsilon_t$$

Then if we solve this equation for ε_t, we find

$$\varepsilon_t = \frac{y_t - (L_{t-1} + B_{t-1})SN_{t-L}}{(L_{t-1} + B_{t-1})SN_{t-L}}$$

If we substitute the right side of this equation for ε_t in the three state equations for the multiplicative Holt–Winters method, we obtain

$$L_t = L_{t-1} + B_{t-1} + \alpha \frac{[y_t - (L_{t-1} + B_{t-1})SN_{t-L}]}{SN_{t-L}}$$

$$B_t = B_{t-1} + \alpha\gamma \frac{[y_t - (L_{t-1} + B_{t-1})SN_{t-L}]}{SN_{t-L}}$$

$$SN_t = SN_{t-L} + (1 - \alpha)\delta \frac{[y_t - (L_{t-1} + B_{t-1})SN_{t-L}]}{L_{t-1} + B_{t-1}}$$

With one minor change we can see that the form of these equations is the same as the error correction form if the estimates replace the true values. The minor change is in the smoothing equation for the seasonal factor, where the state space model requires that the revised values must depend on past time periods. Hence, we see a divisor of $L_{t-1} + B_{t-1}$ instead of L_t.

As stated at the beginning of the section, the models are needed to derive formulas for the prediction intervals. One can also check that the models give us the point forecasts for the corresponding exponential smoothing methods. In the following example we see how the point forecasts and the prediction intervals for simple exponential smoothing are based on the model.

EXAMPLE 8.6

Assume that we have observed the values $y_1, y_2, \ldots, y_T$ for the first T time periods. If we have perfect information so that we know the values of L_0 and α, then by using the observation equation and state equation repeatedly, we would know the value of L_T. Now the mean or expected value of $y_{T+\tau}$, which is τ periods in the future, is found as follows:

If $\tau = 1$ $\quad E(y_{T+1}) = E(L_T + \varepsilon_{T+1}) = E(L_T) + E(\varepsilon_{T+1}) = L_T + 0 = L_T$

If $\tau = 2$ $\quad E(y_{T+2}) = E(L_{T+1} + \varepsilon_{T+2}) = E(L_T + \alpha\varepsilon_{T+1} + \varepsilon_{T+2})$

$$= E(L_T) + \alpha E(\varepsilon_{T+1}) + E(\varepsilon_{T+2}) = L_T$$

If $\tau = 3$ $\quad E(y_{T+3}) = E(L_{T+2} + \varepsilon_{T+3}) = E(L_{T+1} + \alpha\varepsilon_{T+2} + \varepsilon_{T+3})$

$$= E(L_T + \alpha\varepsilon_{T+1} + \alpha\varepsilon_{T+2} + \varepsilon_{T+3}) = L_T$$

and in general,

$$E(y_{T+\tau}) = E(L_T + \alpha\varepsilon_{T+1} + \alpha\varepsilon_{T+2} + \cdots + \alpha\varepsilon_{T+\tau-1} + \varepsilon_{T+\tau}) = L_T$$

Our best forecast for $y_{T+\tau}$ is its expected or mean value. Since we do not have perfect information, we use the estimate ℓ_T for L_T. Hence, the point forecast for $y_{T+\tau}$ is

$$\hat{y}_{T+\tau}(T) = \ell_T$$

A 95% prediction interval for $y_{T+\tau}$ is

$$\hat{y}_{T+\tau}(T) \pm z_{[.025]}\sqrt{\text{Var}(y_{T+\tau} - \hat{y}_{T+\tau}(T))}$$

Again assuming that we have perfect information and thus know the value of L_T, we can find a formula for $\text{Var}(y_{T+\tau} - \hat{y}_{T+\tau}(T)) = \text{Var}(y_{T+\tau} - L_T)$ as follows:

$$\text{If } \tau = 1 \quad \text{Var}(y_{T+1} - L_T) = \text{Var}(L_T + \varepsilon_{T+1} - L_T) = \text{Var}(\varepsilon_T) = \sigma^2$$

$$\begin{aligned}\text{If } \tau = 2 \quad \text{Var}(y_{T+2} - L_T) &= \text{Var}(L_{T+1} + \varepsilon_{T+2} - L_T)\\ &= \text{Var}(L_T + \alpha\varepsilon_{T+1} + \varepsilon_{T+2} - L_T)\\ &= \text{Var}(\alpha\varepsilon_{T+1} + \varepsilon_{T+2})\\ &= \alpha^2\text{Var}(\varepsilon_{T+1}) + \text{Var}(\varepsilon_{T+2})\\ &= \alpha^2\sigma^2 + \sigma^2 = \sigma^2(\alpha^2 + 1)\end{aligned}$$

$$\begin{aligned}\text{If } \tau = 3 \quad \text{Var}(y_{T+3} - L_T) &= \text{Var}(L_{T+2} + \varepsilon_{T+3} - L_T)\\ &= \text{Var}(L_{T+1} + \alpha\varepsilon_{T+2} + \varepsilon_{T+3} - L_T)\\ &= \text{Var}(L_T + \alpha\varepsilon_{T+1} + \alpha\varepsilon_{T+2} + \varepsilon_{T+3} - L_T)\\ &= \text{Var}(\alpha\varepsilon_{T+1} + \alpha\varepsilon_{T+2} + \varepsilon_{T+3})\\ &= \alpha^2\text{Var}(\varepsilon_{T+1}) + \alpha^2\text{Var}(\varepsilon_{T+2}) + \text{Var}(\varepsilon_{T+3})\\ &= \alpha^2\sigma^2 + \alpha^2\sigma^2 + \sigma^2 = \sigma^2(2\alpha^2 + 1)\end{aligned}$$

and in general,

$$\begin{aligned}\text{Var}(y_{T+\tau} - L_T) &= \text{Var}(L_T + \alpha\varepsilon_{T+1} + \alpha\varepsilon_{T+2} + \cdots + \alpha\varepsilon_{T+\tau-1} + \varepsilon_{T+\tau} - L_T)\\ &= \alpha^2\text{Var}(\varepsilon_{T+1}) + \alpha^2\text{Var}(\varepsilon_{T+2}) + \cdots + \alpha^2\text{Var}(\varepsilon_{T+\tau-1}) + \text{Var}(\varepsilon_{T+\tau})\\ &= (\tau - 1)\alpha^2\sigma^2 + \sigma^2 = \sigma^2[(\tau - 1)\alpha^2 + 1]\end{aligned}$$

Since the standard error s is an estimate of σ, a 95% prediction interval for $y_{T+\tau}$ is

$$\ell_T \pm z_{[.025]}\sqrt{s^2[(\tau - 1)\hat{\alpha}^2 + 1]}$$

or

$$\ell_T \pm z_{[.025]}s\sqrt{(\tau - 1)\hat{\alpha}^2 + 1}$$

This is the formula for the 95% prediction interval that was given in Section 8.1.

In a manner similar to Example 8.6, all the point forecasts and prediction intervals for the exponential smoothing methods can be derived from the state space models of Table 8.3. These formulas are analytical formulas. We can also find the prediction intervals by using the model to simulate future values of the time series. We use the cod catch data of Examples 8.1 and 8.2 to illustrate how to simulate the lower and upper limits of a 95% prediction interval.

EXAMPLE 8.7 In order to find the 95% prediction interval for the cod catch data by using simulation, we begin by simulating the future values of the cod catch data. To simulate the future values of the cod catch data at time periods 25, 26, and 27, we use the state space model for simple exponential smoothing as follows:

1. We assume that the final estimates in Figure 8.2(b) are the true values for the model. We assume that $\ell_{24} = 354.5438$, $\sigma = 34.95$, and $\alpha = .034$.
2. We must randomly select values for the error terms ε_{25}, ε_{26}, and ε_{27} from a normal distribution with mean 0 and standard deviation $\sigma = 34.95$.
3. A value for a future cod catch in time period 25 would be found with the observation equation

$$y_{25} = \ell_{24} + \varepsilon_{25} = 354.5438 + \varepsilon_{25}$$

4. A value for a future cod catch in time period 26 would be found by using the state equation

$$\ell_{25} = \ell_{24} + \alpha\varepsilon_{25} = 354.5438 + (.034)\varepsilon_{25}$$

and then the observation equation

$$y_{26} = \ell_{25} + \varepsilon_{26}$$

5. A value for a future cod catch in time period 27 would be found by using the state equation

$$\ell_{26} = \ell_{25} + \alpha\varepsilon_{26} = \ell_{25} + (.034)\varepsilon_{26}$$

and then the observation equation

$$y_{27} = \ell_{26} + \varepsilon_{27}$$

Note: If one is given values for ε_{25}, ε_{26}, and ε_{27}, the equations in steps 3, 4, and 5 can readily be used to compute y_{25}, y_{26}, and y_{27} (see Exercise 8.23).

If we generate many values for ε_{25}, ε_{26}, and ε_{27}, say 10,000 of each, we would have many values for y_{25}, y_{26}, and y_{27} (10,000 values of each). Then we choose

$$LL95_{25} = \text{2.5th percentile of the 10,000 values for } y_{25}$$
$$UL95_{25} = \text{97.5th percentile of the 10,000 values for } y_{25}$$

The 95% prediction interval for y_{25} is

$$[LL95_{25}, UL95_{25}]$$

Similarly, the 95% prediction intervals for y_{26} and y_{27} are

$$[LL95_{26}, UL95_{26}] \quad \text{and} \quad [LL95_{27}, UL95_{27}]$$

We demonstrate this process with Crystal Ball, a simulation add-in for Excel. Table 8.4 shows where the formulas would be entered into the Excel spreadsheet of Figure 8.2(b) for the simulation of y_{25}, y_{26}, and y_{27}. The results of the simulation are shown in Figure 8.16. The medians should be close to the means in a simulated normal distribution, and hence the medians (50.0th percentiles) should be close to the point forecasts of 354.5438 (or 355) in Example 8.2. The medians for time periods 25, 26, and 27 are 354.8518, 353.94, and 354.33 (or 355, 354, and 354), respectively.

From Figure 8.16, we see that the 95% prediction intervals are

[285.7176, 422.6914] or [286, 423] for y_{25}

[284.37, 424.47] or [284, 424] for y_{26}

and

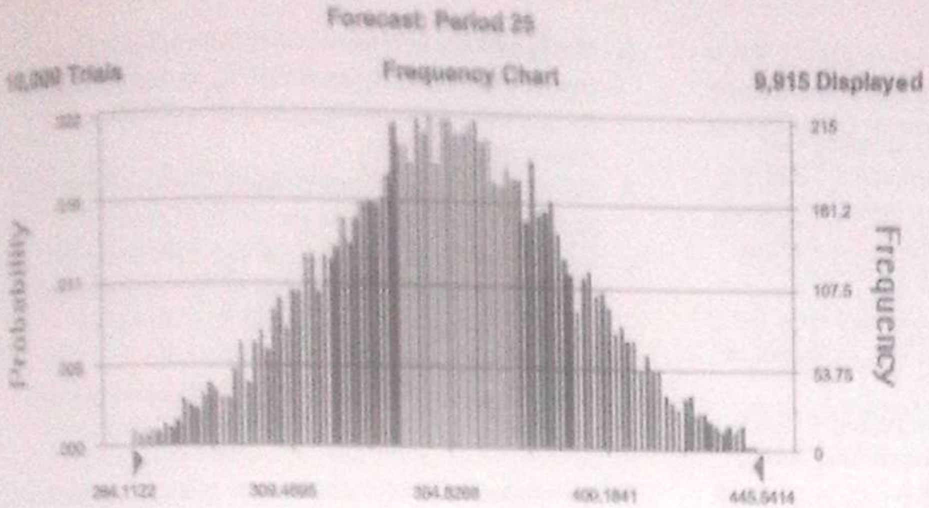
[284.54, 422.26] or [285, 422] for y_{27}

These intervals should be compared with the analytical intervals in Example 8.2. Although the 95% prediction intervals that were derived analytically are more precise statistically, the simulated intervals are very close. Simulating intervals has some advantages. We can find intervals for which we do not have analytical formulas. For example, we can readily simulate 95% intervals for the cumulative cod catch for all three months of the cod catch data and for time series when the multiplicative Holt–Winters method is appropriate. In addition, the distribution for future values of a time series for which the multiplicative Holt–Winters method is appropriate may not have a normal distribution even if the ε_t values are normally distributed. Thus simulated prediction intervals may be the most reasonable intervals.

TABLE 8.4 Entries for Simulation of y_{25}, y_{26}, and y_{27} for Cod Catch

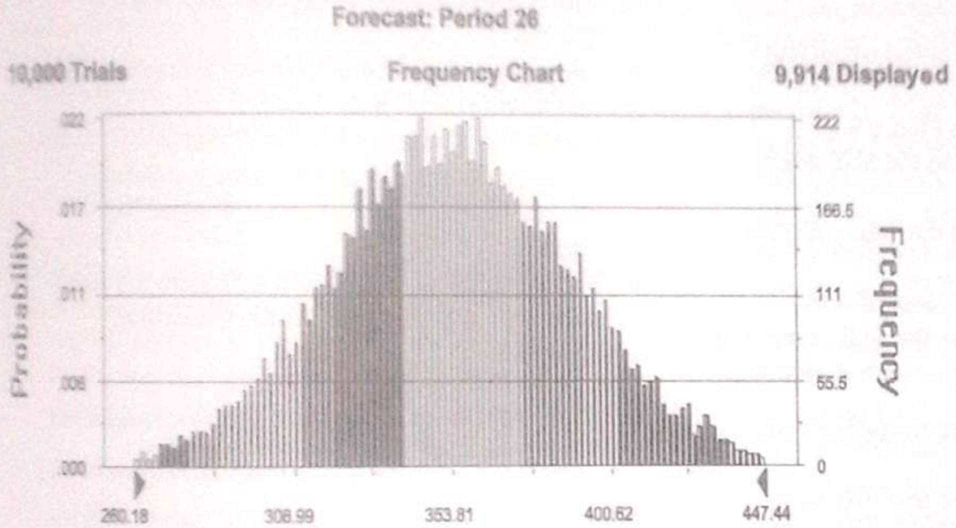
n	alpha	SSE	ssquare	s	
24	0.0343532	28089	1221.27	34.95	
Time Period	**Actual Cod Catch y**	**Smoothed Estimate for Level**	**Forecast Made Last Period**	**Forecast Error**	**Squared Forecast Error**
0		360.6667			
1	362	360.7125	360.6667	1.3333	1.7778
2	381	361.4094	360.7125	20.2875	411.5838
3	317	359.8838	361.4094	−44.4094	1972.1959
.		.			.
.		.			.
.		.			.
21	345	355.3733	355.7424	−10.7424	115.3987
22	362	355.6010	355.3733	6.6267	43.9126
23	314	354.1719	355.6010	−41.6010	1730.6423
24	365	354.5438	354.1719	10.8281	117.2486
Simulation					
25	$y_{25} = L_{24} + \varepsilon_{25}$	$L_{25} = L_{24} + \alpha\varepsilon_{25}$		ε_{25}	
26	$y_{26} = L_{25} + \varepsilon_{26}$	$L_{26} = L_{25} + \alpha\varepsilon_{26}$		ε_{26}	
27	$y_{27} = L_{26} + \varepsilon_{27}$			ε_{27}	

FIGURE 8.16 Crystal Ball results when forecasting y_{25}, y_{26}, and y_{27} for the cod catch



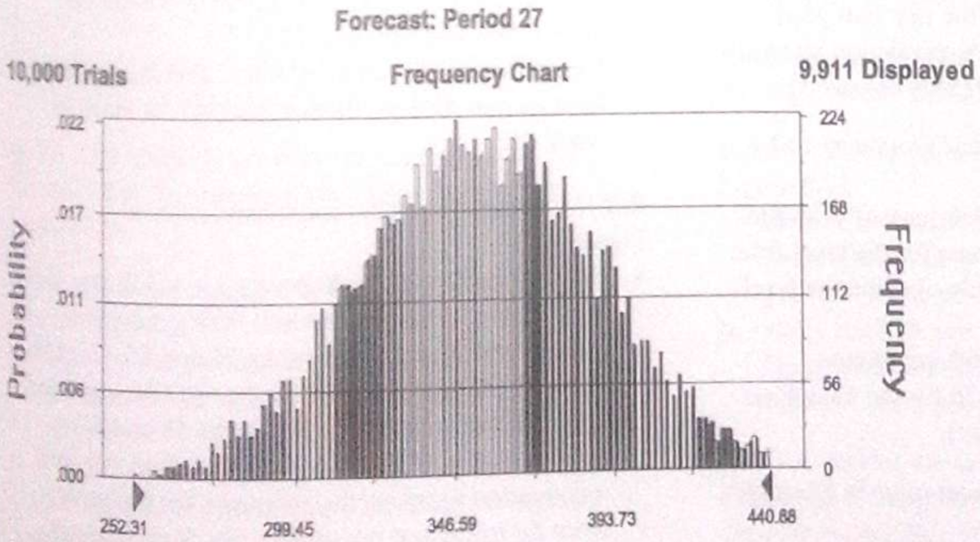
Percentile	Value
0.0%	221.6851
2.5%	285.7176
5.0%	296.6148
50.0%	354.8518
95.0%	411.9697
97.5%	422.6914
100.0%	493.5687

(a) Period 25



Percentile	Value
0.0%	209.21
2.5%	284.37
5.0%	295.25
50.0%	353.94
95.0%	413.04
97.5%	424.47
100.0%	482.73

(b) Period 26



Percentile	Value
0.0%	200.47
2.5%	284.54
5.0%	296.28
50.0%	354.33
95.0%	411.96
97.5%	422.26
100.0%	518.21

(c) Period 27

Exercises

8.1 Consider the Bay City Seafood Company cod catch data that were analyzed in Examples 8.1 and 8.2.

a. Verify that ℓ_3, an estimate made in period 3 (March of year 1) of the level of the cod catch time series is 358.2380, as shown in Figure 8.1.
b. Verify that the one-period-ahead forecast error for period 4 (April of year 1) is −61.2380, as shown in Figure 8.1.
c. Verify that ℓ_4, an estimate made in period 4 (April of year 1) of the level of the cod catch time series, is 352.1142, as shown in Figure 8.1.
d. Verify that the one-period-ahead forecast error for period 5 (May of year 1) is 46.8858, as shown in Figure 8.1.

8.2 Consider the Bay City Seafood Company cod catch data in Figure 8.1.

a. Set up the Excel spreadsheet in Figure 8.1.
b. Use the Excel spreadsheet to find the SSE when $\alpha = .4$.
c. Use trial and error to try to find the value of α that minimizes the SSE.
d. Use Solver in Excel to find the value of α that produces the minimum value for the SSE. Your resulting spreadsheet should agree with Figure 8.2.

8.3 Consider the Bay City Seafood Company cod catch data in Figure 8.2.

a. Using the observed values of the first two years (that is, use $T = 24$), find the point forecast and 95% prediction interval for the cod catch in month 28 (April of year 3).
b. Using the observed values of the first two years, find the point forecast and 95% prediction interval for the cod catch in month 29 (May of year 3).

8.4 Consider the Bay City Seafood Company cod catch data in Figure 8.2.

a. If we observe a cod catch in February of year 3 to be $y_{26} = 375$, update the estimate for the level from ℓ_{25} to ℓ_{26}. Recall that we already updated the level to ℓ_{25} in Example 8.2.
b. Find the point forecasts and 95% prediction intervals made in time period 26 for the next three months (March, April, and May).

8.5 Consider the weekly thermostat sales in Example 8.3.

a. Verify that ℓ_2, an estimate for the level made in period 2, is 211.1694 and that b_2, an estimate for the growth rate made in period 2, is .5524, as shown in Figure 8.6.
b. Verify that the one-period-ahead forecast error in period 3 is −26.7219, as shown in Figure 8.6.
c. Verify that ℓ_3, an estimate for the level made in period 3, is 206.3775 and that b_3, an estimate for the growth rate made in period 3, is .0180, as shown in Figure 8.6.
d. Verify that the one-period-ahead forecast error in period 4 is −37.3955, as shown in Figure 8.6.

8.6 Consider the weekly thermostat sales in Figure 8.6.

a. Set up the Excel spreadsheet in Figure 8.6 with $\ell_0 = 202.6246$ and $b_0 = -.3682$.
b. Use this spreadsheet to find the SSE when $\alpha = .1$ and $\gamma = .1$.
c. Use trial and error to try to find the α and γ values that minimize the SSE.
d. Use Solver, starting with $\alpha = .2$ and $\gamma = .1$, to find the values of α and γ that produce a minimum value for SSE. The results should agree with Figure 8.7.
e. Set up the Excel spreadsheet in Figure 8.6 using the error correction form of the smoothing equations.

8.7 Use Excel to produce the regression output for the thermostat sales as shown in Figure 8.6.

8.8 Consider the weekly thermostat sales in Figure 8.7(b).

a. Using the first 52 weeks of sales (that is, use $T = 52$), find the point forecast and 95% prediction interval for sales in week 56.
b. Using the first 52 weeks of sales, find the point forecast and 95% prediction interval for sales in week 57.

8.9 Consider the weekly thermostat sales in Figure 8.7(b).

a. In Example 8.3, after observing $y_{53} = 330$ we revised the point forecasts and 95% prediction intervals for the sales in weeks 54 and 55. Continue this revision by finding revised point forecasts and 95% prediction intervals for weeks 56 and 57.
b. Suppose we now observe $y_{54} = 320$. Use this new information to revise the estimates for the level from ℓ_{53} to ℓ_{54} and the growth rate from b_{53} to b_{54}.

c. Using the new estimates from part (b), revise the point forecasts and 95% prediction intervals for the sales in weeks 55, 56, and 57.

8.10 Consider the calculator sales in Table 6.2 and Figure 6.4 of Chapter 6.

a. Find the initial values ℓ_0 and b_0 by using Excel or some other statistical package to fit a straight line to the first half of the data.
b. Set up an Excel spreadsheet to perform Holt's trend corrected exponential smoothing.
c. Use Solver to find the values of α and γ that minimize the SSE.
d. Find point forecasts and 95% prediction intervals for January, February, and March of year 3.

8.11 Consider the sales of the TRK-50 mountain bike in Example 8.4.

a. Verify the following estimates for the level, growth rate, and seasonal factor: $\ell_2 = 23.5586$, $b_2 = 1.0508$, and $sn_2 = 6.6418$, as shown in Figure 8.9.
b. Verify that the one-period-ahead forecast error in period 3 is $-.1815$.
c. Verify the following estimates for the level, growth rate, and seasonal factor: $\ell_3 = 24.5731$, $b_3 = 1.0472$, and $sn_3 = 18.5575$.
d. Verify that the one-period-ahead forecast error in period 4 is 1.2885.

8.12 Consider the sales of the TRK-50 mountain bike in Figure 8.9.

a. Set up the Excel spreadsheet in Figure 8.9 using $\ell_0 = 20.8500$, $b_0 = .9809$, $sn_{-3} = -14.2162$, $sn_{-2} = 6.5529$, $sn_{-1} = 18.5721$, and $sn_0 = -10.9088$.
b. Use Solver in your Excel spreadsheet from part (a) to verify that $\alpha = .561$, $\gamma = 0$, and $\delta = 0$ minimize the SSE, as shown in Figure 8.10.

8.13 Use Excel to produce the regression output for the TRK-50 mountain bike sales as shown in Figure 8.9.

8.14 Consider the sales of the TRK-50 mountain bike in Figure 8.10.

a. Find a point forecast and 95% prediction interval for sales of the mountain bike in the fourth quarter of year 5.
b. Find a point forecast and 95% prediction interval for sales of the mountain bike in the first quarter of year 6.

8.15 Consider the sales of Tiger Sports Drink in Example 8.5.

a. Using the regression estimates in Figure 8.13, compute S_2, S_6, S_{10}, and S_{14}.
b. Compute $\bar{S}_{[2]}$.
c. Verify that the initial estimate of the seasonal factor for quarter 2 is $sn_{-2} = 1.1114$, as shown in Figures 8.12 and 8.13.
d. Repeat the process in parts (a) through (c) to verify that the initial estimate of the seasonal factor for quarter 3 is $sn_{-1} = 1.2937$, as shown in Figure 8.12.

8.16 Consider the sales of the Tiger Sports Drink in Example 8.5.

a. Verify the following estimates for the level, growth rate, and seasonal factor: $\ell_3 = 104.5393$, $b_3 = 2.6349$, $sn_3 = 1.2944$, as shown in Figure 8.12.
b. Verify that the one-period-ahead forecast error in period 4 is .7650.
c. Verify the following estimates for the level, growth rate, and seasonal factor: $\ell_4 = 107.3464$, $b_4 = 2.6521$, $sn_4 = .8892$, as shown in Figure 8.12.
d. Verify that the one-period-ahead forecast error in period 5 is $-.9479$.

8.17 Consider the sales of the Tiger Sports Drink in Figure 8.12.

a. Set up the Excel spreadsheet in Figure 8.12 using $\ell_0 = 95.2500$, $b_0 = 2.4706$, $sn_{-3} = .7062$, $sn_{-2} = 1.1114$, $sn_{-1} = 1.2937$, and $sn_0 = .8886$.
b. Use Solver, starting with $\alpha = .2$, $\gamma = .1$, and $\delta = .1$ in the Excel spreadsheet from part (a), to verify the values $\alpha = .336$, $\gamma = .046$, and $\delta = .134$ minimize the SSE, as shown in Figure 8.14(b).

8.18 Use Excel to produce the regression output for the Tiger Sports Drink as shown in Figure 8.13.

8.19 Consider the sales of Tiger Sports Drink in Figure 8.14.

a. Verify the point forecasts of the sales of Tiger Sports Drink for quarter 3 (time period 35) and quarter 4 (time period 36) as given in Example 8.5.
b. Verify the 95% prediction intervals for the sales of Tiger Sports Drink for quarter 3 (time period 35) and quarter 4 (time period 36) as given in Example 8.5.

8.20 Consider the sales of Tiger Sports Drink in Figure 8.14. Suppose in the first quarter of year 9, we observe $y_{33} = 124$.

a. Without finding new values for the smoothing constants, find the estimates ℓ_{33}, b_{33}, and sn_{33} for the level, growth rate, and seasonal factor in time period 33.
b. Use the estimates from part (a) to revise the point forecasts and 95% prediction intervals for the sales of the sports drink in time periods 34, 35, and 36.

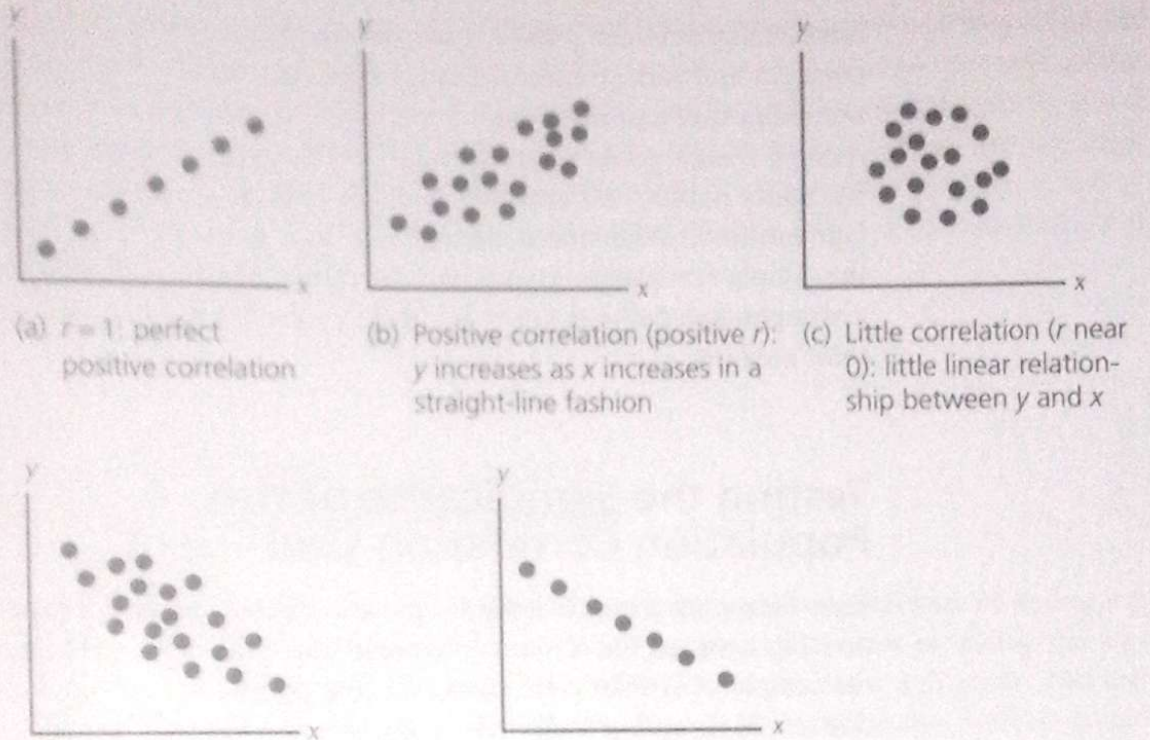
8.21 Consider the Tasty Cola data of Table 7.1 and Figure 8.15. Use MINITAB to find smoothing constants that produce a smaller SSE than in Figure 8.15, where the SSE = (6812.61)(36) = 245,253.96.

8.22 Consider the Tasty Cola data of Table 7.1.
a. Set up an Excel spreadsheet to find smoothing constants that minimize the SSE.
b. Use regression to fit a straight line to the Tasty Cola data and find initial estimates for the level and growth rate.
c. Use the procedures of Section 8.4 to find initial values for the 12 seasonal factors.
d. Use Solver in Excel to find smoothing constants that minimize the SSE.
e. Compute point forecasts and prediction intervals for the first four months of the fourth year.

8.23 Consider the cod catch data in Example 8.7.
a. If the values for ε_{25}, ε_{26}, and ε_{27} are 20, −15, and −5, respectively, simulate by hand the values for y_{25}, y_{26}, and y_{27}.
b. If the values of ε_{25}, ε_{26}, and ε_{27} are −30, 4, and 22, respectively, simulate by hand the values for y_{25}, y_{26}, and y_{27}.

8.24 Use Crystal Ball or some other simulation add-in for Excel to simulate the 95% prediction intervals for the future values of y_{53}, y_{54}, y_{55}, y_{56}, and y_{57} for the weekly thermostat sales data and compare the results with the analytical prediction intervals in Example 8.3 and Exercise 8.8.

FIGURE 3.14 An illustration of different values of the simple correlation coefficient



(a) $r = 1$: perfect positive correlation

(b) Positive correlation (positive r): y increases as x increases in a straight-line fashion

(c) Little correlation (r near 0): little linear relationship between y and x

(d) Negative correlation (negative r): y decreases as x increases in a straight-line fashion

(e) $r = -1$; perfect negative correlation

If we have computed the least squares slope b_1 and r^2, the method given in the previous box provides the easiest way to calculate r. The simple correlation coefficient can also be calculated using the formula

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

Here SS_{xy} and SS_{xx} have been defined in Section 3.2, and SS_{yy} denotes the total variation, which has been defined in this section. Furthermore, this formula for r automatically gives r the correct (+ or −) sign. For instance, in the fuel consumption problem, $SS_{xy} = -179.6475$, $SS_{xx} = 1404.355$, and $SS_{yy} = 25.549$ (see Table 3.3 and Figure 3.9). Therefore

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}} = \frac{-179.6475}{\sqrt{(1404.355)(25.549)}} = -.948$$

It is important to make a couple of points. First, *the value of the simple correlation coefficient is not the slope of the least squares line*. If we wish to find this slope, we should use the previously given formula for b_1. Second, *high correlation does not imply that a cause-and-effect relationship exists*. When r indicates that y and x are highly correlated, this says that y and x have a strong tendency to move together in a straight-line fashion. The correlation does not mean that changes in x cause changes in y. Instead, some other variable (or variables) could be causing the apparent