Tutorial letter 201/2/2017

LINEAR ALGEBRA

MAT2611

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 01.

BARCODE

Question 1

Consider the set

$$X := \left\{ \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \;:\; a \in \mathbb{R} \right\} \subset M_{22}$$

and the operations $\left(\text{for all } k, a, b \in \mathbb{R},\ \mathbf{u} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X \text{ and } \mathbf{v} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X\right)$

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{u} \equiv k \cdot \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} := \begin{bmatrix} ka & 1 \\ 0 & -ka \end{bmatrix},$$

$$+ : X \times X \to X, \qquad \mathbf{u} + \mathbf{v} \equiv \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & 1 \\ 0 & -b \end{bmatrix} := \begin{bmatrix} a+b & 1 \\ 0 & -(a+b) \end{bmatrix}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements are true in this vector space?

1. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$
2. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$
3. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
4. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$
5. None of the above.

Answer: 4

Since $-\mathbf{u} = (-1) \cdot \mathbf{u}$ *for all* $\mathbf{u} \in X$

$$-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = (-1) \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} (-1) \cdot 1 & 1 \\ 0 & -(-1) \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Alternative:

Let $\mathbf{0} = \begin{bmatrix} z & 1 \\ 0 & -z \end{bmatrix} \in X$*. Then* $\mathbf{0}$ *satisfies for all* $\mathbf{u} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X$

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} z & 1 \\ 0 & -z \end{bmatrix} + \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} = \begin{bmatrix} z+a & 1 \\ 0 & -(z+a) \end{bmatrix} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} = \mathbf{u}$$

so that $z + a = a$ *(and* $1 = 1$, $0 = 0$, $-(z+a) = -a$*). It follows that* $z = 0$ *so that*

$$\mathbf{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Let $-\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix} = \begin{bmatrix}n & 1\\0 & -n\end{bmatrix} \in X$ *which satisfies*

$$\begin{bmatrix}n & 1\\0 & -n\end{bmatrix} + \begin{bmatrix}1 & 1\\0 & -1\end{bmatrix} = \begin{bmatrix}n+1 & 1\\0 & -(n+1)\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

so that $n+1=0$. *It follows that* $n=-1$ *so that*

$$-\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 1\\0 & 1\end{bmatrix}.$$

Question 2

Which of the following are subspaces of P_2 with the usual operations ?

A. $\operatorname{span}\{1, x^2\}$

B. $\{1+ax : a \in \mathbb{R}\}$

C. $\{a-bx^2 : a, b \in \mathbb{R}\}$

D. $\{a : a \in \mathbb{R}, a \geq 0\}$

Select from the following:

1. Only A, B and C.
2. Only A, C and D.
3. Only C and D.
4. Only A and C.
5. None of the above.

Answer: 4

The span of any subset of a vector space is by definition a vector subspace of that space. Since $1, x^2 \in P_2$, $\operatorname{span}\{1, x^2\}$ *is a subspace of* P_2.

The remaining sets are subpspaces of P_2 *if they are non-empty subsets of* P_2 *and are closed with respect to vector addition and scalar multiplication.*

Since $1 \in \{1+ax : a \in \mathbb{R}\}$, *this set is non-empty. However,* $(1+ax)+(1+bx) = 2+(a+b)x \notin \{1+ax : a \in \mathbb{R}\}$.

Since $0 \in \{a-bx^2 : a, b \in \mathbb{R}\}$, *this set is non-empty. We also have* $(a_1-b_1x^2)+(a_2-b_2x^2) = (a_1+a_2)-(b_1+b_2)x^2 \in \{a-bx^2 : a, b \in \mathbb{R}\}$ *for* $a_1, a_2, b_1, b_2 \in \mathbb{R}$. *We also have* $k(a-bx^2) = (ka)-(kb)x^2 \in \{a-bx^2 : a, b \in \mathbb{R}\}$ *for* $k, a, b \in \mathbb{R}$. *Thus this set with the usual operations in* P_2 *is a subspace of* P_2.

Since $a \in \{a : a \in \mathbb{R}, a \geq 0\}$, *this set is non-empty. However,* $1 \in \{a : a \in \mathbb{R}, a \geq 0\}$ *but* $(-1)\cdot 1 = -1 \notin \{a : a \in \mathbb{R}, a \geq 0\}$. *In other words, the set is not closed under scalar multiplication.*

Question 3

Which of the following sets are linearly independent?

A. $\{ (1,0), (1,1), (1,-1) \}$ in $\mathbb{R}^2$

B. $\{ (1,1,1), (1,-1,1), (2,-3,2) \}$ in $\mathbb{R}^3$

C. $\{ 1+x, x, 2+3x \}$ in P_2

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A, B and C.
2. Only B and C.
3. Only B and D.
4. Only D.
5. None of the above.

Answer: 4

Since $\dim(\mathbb{R}^2) = 2$, *a set of 3 vectors in* $\mathbb{R}^2$ *cannot be linearly independent (since* $3 > 2$*). We can also see this from the equation*

$$c_1(1,0) + c_2(1,1) + c_3(1,-1) = (c_1 + c_2 + c_3, c_2 - c_3) = (0,0)$$

which has the non-trivial solution $c_1 = -2$ *and* $c_2 = c_3 = 1$. *The set A is linearly dependent.*

From the equation

$$c_1(1,1,1) + c_2(1,-1,1) + c_3(2,-3,2) = (c_1 + c_2 + 2c_3, c_1 - c_2 - 3c_3, c_1 + c_2 + 2c_3) = (0,0,0)$$

we obtain two equations $c_1 + c_2 + 2c_3 = c_1 - c_2 - 3c_3 = 0$ *for three variables* c_1, c_2, c_3. *Since we know the equation has the solution* $c_1 = c_2 = c_3 = 0$, *the equation also has infinitely many non-trivial solutions. For example,* $c_1 = 1$, $c_2 = -5$ *and* $c_3 = 2$ *is a non-trivial solution of the equation. The set B is linearly dependent.*

The set C is clearly a linearly dependent set since $2 + 3x = 2(1 + x) + x$, *i.e. the equation*

$$c_1(1 + x) + c_2 x + c_3(2 + 3x) = 0$$

has the non-trivial solution $c_1 = 2$, $c_2 = 1$ *and* $c_3 = -1$.

The set D is linearly independent since

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & c_1 - c_2 \\ c_1 + c_2 & c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

if and only if $c_1 + c_2 = c_1 - c_2 = 0$ *if and only if* $c_1 = c_2 = 0$ *(since* $c_1 = c_2$ *the result follows immediately).*

Question 4

Which of the following sets are a basis for the following vector subspace of M_{22}:

$$X = \left\{ A \in M_{22} \,:\, A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

A. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A and B.
2. Only B and C.
3. Only C and D.
4. Only A and D.
5. None of the above.

Answer: 3

The matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in X$ *satisfy*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $a = b$ *and* $c = d$ *and*

$$X = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} \,:\, a, c \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Since

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is a linearly independent set, the set C *is a basis for* X*. Since every basis for a finite dimensional vector space must have the same number of elements, sets* A *and* C *are not bases for* X*. Now we* *consider set D. The set* $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ *is linearly independent (check this!) and*

$$\begin{aligned} \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\} &= \left\{ a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a+b & a+b \\ b & b \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} c & c \\ d & d \end{bmatrix} \ : \ c, d \in \mathbb{R} \right\} && \textit{(set } c = a+b \text{ and } d = b\textit{)} \\ &= X \end{aligned}$$

The second last step follows since $b = d$ and $a = c - d$ are uniquely determined by c and d and vice-versa. Thus X is spanned by the linearly independent set D and the set D is also a basis for X.

Question 5

Which of the following statements are true:

A. $\dim(\operatorname{span}\{ 1 + x^2, 1 - x^2 \}) = 2$ in P_2

B. $\dim(\operatorname{span}\{ x^2, -x^2 \}) = 2$ in P_2

C. $\dim(\operatorname{span}\{ 1 + x + x^2, 1 + x - x^2, 1 - x + x^2, -1 + x + x^2 \}) = 4$ in P_2

Select from the following:

1. All of A, B, and C.
2. Only A and C.
3. Only A and B.
4. Only A.
5. None of the above.

Answer: 4

Since $\{ 1 + x^2, 1 - x^2 \}$ *is a linearly independent set,* $\dim(\operatorname{span}\{ 1 + x^2, 1 - x^2 \}) = 2$*. Since*

$$\operatorname{span}\left\{ x^2, -x^2 \right\} = \operatorname{span}\left\{ x^2 \right\}$$

and $\{ x^2 \}$ *is a linearly independent set,* $\dim(\operatorname{span}\{ x^2, -x^2 \}) = 1$*. Finally, no subspace of P_2 has dimension greater than 3 (since* $\dim(P_2) = 3$*) and statement C is false.*

Question 6

Which of the following sets are a basis for the column space of $\begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Select from the following:

1. All of A, B, C and D.
2. Only B, C and D.
3. Only A.
4. Only B and C.
5. None of the above.

Answer: 2

A basis for the column space is a linearly independent set such that every column of the matrix can be expressed as a linear combination of the elements of the set and vice-versa. The set in A is not linearly independent since

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The set in B is linearly independent and

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Thus the set in B is a basis for the column space. The set in C is linearly independent and

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Thus the set in C is a basis for the column space. The set in D is linearly independent and

$$\begin{bmatrix}1\\3\end{bmatrix} = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\end{bmatrix}, \quad \begin{bmatrix}3\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\end{bmatrix},$$

$$\begin{bmatrix}1\\0\end{bmatrix} = -1/2\begin{bmatrix}1\\3\end{bmatrix} + 3/2\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}3\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\end{bmatrix} = 0\begin{bmatrix}1\\3\end{bmatrix} + 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}3\\1\end{bmatrix}.$$

Thus the set in D is a basis for the column space.

Question 7

Which of the following sets are a basis for the null space of $\begin{bmatrix}1 & 1 & -1\\0 & -1 & 1\end{bmatrix}$?

A. $\left\{ \begin{bmatrix}0 & 1 & 1\end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix}0 & 1 & 1\end{bmatrix}^T, \begin{bmatrix}2 & -1 & 1\end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix}1 & 1 & -1\end{bmatrix}^T, \begin{bmatrix}0 & -1 & 1\end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix}1 & 0\end{bmatrix}^T, \begin{bmatrix}1 & -1\end{bmatrix}^T \right\}$

Select from the following:

1. Only A.
2. Only C.
3. Only B.
4. Only A.
5. None of the above.

Answer: 1

Elements $\begin{bmatrix}a & b & c\end{bmatrix}^T$ *of the null space satisfy*

$$\begin{bmatrix}1 & 1 & -1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a-(c-b)\\c-b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Clearly $a = 0$ *and* $b = c$. *Thus the null space is*

$$N\left(\begin{bmatrix}1 & 1 & -1\\0 & -1 & 1\end{bmatrix}\right) = \left\{ \begin{bmatrix}0\\b\\b\end{bmatrix} : b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}.$$

The dimension of the null space is 1, so that every basis must consist of only 1 vector. Obviously set A is a basis for the null space, and sets B and C are not bases for the null space (too many elements) while set D makes no sense (incompatible with the matrix multiplication).

Question 8

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrices A ?

A. $\text{rank}(A) = \text{rank}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A^T) = m$

C. $\text{rank}(A^T) + \text{nullity}(A^T) = n$

D. $\text{row space}(A) = \text{column space}(A)$

Select from the following:

1. Only A and B.
2. Only A and C.
3. Only C.
4. Only A.
5. None of the above.

Answer: 1