Tutorial letter 202/2/2017

LINEAR ALGEBRA

MAT2611

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 02.

BARCODE

Question 1: 20 Marks

Let

$$B_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \quad \text{and} \quad B_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

be two bases for span(B_1) in M_{22}, where the usual left to right ordering is assumed.

(1.1) Find the transition matrix (change of coordinate/change of basis matrix) $P_{B_1 \to B_2}$. (8)

We express the elements of B_1 in terms of B_2:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + b_1 \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + c_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_1 + c_1 \\ b_1 & -a_1 - b_1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a_1 + b_1 = 1 \\ a_1 + c_1 = 1 \\ b_1 = 1 \\ -a_1 - b_1 = -1 \end{array} \qquad \Rightarrow \begin{array}{l} a_1 = 0 \\ b_1 = 1 \quad \checkmark^2 \\ c_1 = 1 \end{array}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = a_2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + b_2 \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_2 + b_2 & a_2 + c_2 \\ b_2 & -a_2 - b_2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a_2 + b_2 = 0 \\ a_2 + c_2 = 1 \\ b_2 = 1 \\ -a_2 - b_2 = 0 \end{array} \qquad \Rightarrow \begin{array}{l} a_2 = -1 \\ b_2 = 1 \quad \checkmark^2 \\ c_2 = 2 \end{array}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = a_3 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + b_3 \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_3 + b_3 & a_3 + c_3 \\ b_3 & -a_3 - b_3 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} a_3 + b_3 = 0 \\ a_3 + c_3 = -1 \\ b_3 = 1 \\ -a_3 - b_3 = 0 \end{array} \qquad \Rightarrow \begin{array}{l} a_3 = -1 \\ b_3 = 1 \quad \checkmark^2 \\ c_3 = 0 \end{array}$$

Thus we find

$$P_{B_1 \to B_2} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}. \checkmark^2$$

(1.2) Let B_3 be a basis for span(B_1) and let the transition matrix from B_2 to B_3 be given by

$$P_{B_2 \to B_3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Find the transition matrix $P_{B_1 \to B_3}$. (6)

This is a change of basis from B_1 to B_3, which can be achieved by changing basis from B_1 to B_2 and then again changing basis from B_2 to B_3 ($B_1 \to B_3 \equiv B_1 \to B_2 \to B_3$):

$$P_{B_1 \to B_3} = P_{B_2 \to B_3} P_{B_1 \to B_2} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}. \checkmark^6$$

(b) Use $P_{B_2 \to B_3}$ to find B_3. (6)

Since all bases for the same vector space have the same number of elements, B_3 has 3 elements. Suppose $B_3 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, ordered left to right. We have

$$[\mathbf{v}_1]_{B_2} = P_{B_3 \to B_2}[\mathbf{v}_1]_{B_3},$$

and since $[\mathbf{v}_1]_{B_3} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ *and* $P_{B_3 \to B_2} = P^{-1}_{B_2 \to B_3}$

$$[\mathbf{v}_1]_{B_2} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \checkmark$$

This vector provides the expansion coefficients of $\mathbf{v}_1$ in B_2. Thus

$$\mathbf{v}_1 = 1 \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \checkmark$$

Similarly, $[\mathbf{v}_2]_{B_3} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ *so that*

$$[\mathbf{v}_2]_{B_2} = P_{B_3 \to B_2}[\mathbf{v}_2]_{B_3} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \checkmark$$

Thus

$$\mathbf{v}_2 = 1 \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \checkmark$$

Finally, $[\mathbf{v}_3]_{B_3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ *so that*

$$[\mathbf{v}_3]_{B_2} = P_{B_3 \to B_2}[\mathbf{v}_3]_{B_3} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}. \checkmark$$

Thus

$$\mathbf{v}_3 = -1 \cdot \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \checkmark$$

It follows that

$$B_3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Question 2: 20 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2.1) Determine the characteristic equation for A in λ. (4)

The characteristic equation is (cofactor expansion along the 2nd row)

$$\det\left(\lambda\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right) \checkmark = \begin{vmatrix} \lambda-1 & 0 & -1 & 0 \\ 0 & \lambda-1 & 0 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\begin{vmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda & 0 \\ -1 & 0 & \lambda-1 \end{vmatrix}$$

$$= \lambda(\lambda-1)^3. \checkmark^3$$

(2.2) Find the eigenvalues of A, and their algebraic multiplicities. (4)

We solve the characteristic equation $\lambda(\lambda-1)^3 = 0$ for λ, i.e. we find the eigenvalue $\lambda = 0$✓ (algebraic multiplicity 1)✓ and the eigenvalue $\lambda = 1$✓ (algebraic multiplicity 3).✓

(2.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue. (12)

The eigenspace corresponding to the eigenvalue 0 is given by the solutions to the equation

$$\left(0\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

where $a, b, c, d \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix} -1 & 0 & -1 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ -1 & 0 & 0 & -1 & : & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -1 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & -1 & : & 0 \end{bmatrix} \qquad (R_4 \leftarrow R_4 - R_1)$$

$$\to \begin{bmatrix} -1 & 0 & -1 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & -1 & : & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_2)$$

$$\to \begin{bmatrix} -1 & 0 & -1 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_3 \leftrightarrow R_4)$$

$$\to \begin{bmatrix} -1 & 0 & 0 & -1 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_1 \leftarrow R_1 + R_3)$$

so that $a = -d$, $b = 0$, $c = d$ and $d \in \mathbb{R}$ is free. The eigenspace corresponding to the eigenvalue 0 is

$$\left\{ \begin{bmatrix} -d \\ 0 \\ d \\ d \end{bmatrix} : d \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} . \checkmark^2$$

A basis for the eigenspace is given by

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} . \checkmark$$

The geometric multiplicity of the eigenvalue 0 is 1.$\checkmark$

The eigenspace corresponding to the eigenvalue 1 is given by the solutions to the equation

$$\left(1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

where $a, b, c, d \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix} 0 & 0 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & -1 & 1 & 0 & : & 0 \\ -1 & 0 & 0 & 0 & : & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & -1 & 1 & 0 & : & 0 \\ 0 & 0 & -1 & 0 & : & 0 \end{bmatrix} \qquad (R_1 \leftrightarrow R_4)$$

$$\to \begin{bmatrix} -1 & 0 & 0 & 0 & : & 0 \\ 0 & -1 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & -1 & 0 & : & 0 \end{bmatrix} \qquad (R_2 \leftrightarrow R_3)$$

$$\to \begin{bmatrix} -1 & 0 & 0 & 0 & : & 0 \\ 0 & -1 & 1 & 0 & : & 0 \\ 0 & 0 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_3 \leftrightarrow R_4)$$

$$\to \begin{bmatrix} -1 & 0 & 0 & 0 & : & 0 \\ 0 & -1 & 0 & 0 & : & 0 \\ 0 & 0 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2 + R_3)$$

so that $a = b = c = 0$ and d is free. The eigenspace corresponding to the eigenvalue 1 is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} : d \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \right\} . \checkmark^2$$

The dimension of this eigenspace is 1. The geometric multiplicity for the eigenvalue 1 is 1.✓ *A basis for the eigenspace is given by*

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} . \checkmark$$

Note: *when determining the eigenspaces, there will <u>always</u> be <u>at least one free parameter</u>. When solving for the eigenvectors: if you find only $a = b = c = d = 0$, it means that you have made a calculation error — either the eigenvalue is incorrect <u>or</u> the eigenvector calculation is incorrect.*