Tutorial letter 203/2/2017

LINEAR ALGEBRA

MAT2611

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 03.

BARCODE

Define tomorrow.

UNISA university of south africa

Question 1

Let A be an $n \times n$ matrix, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. λ is not an eigenvalue of A.
2. $\lambda = 0$.
3. $\lambda = 0$ and 0 is an eigenvalue of A.
4. A is invertible.
5. None of the above.

Answer: 1

First note that $\mathbf{x} = \mathbf{0}$ *always satisfies the equation. If* λ *is not an eigenvalue of* A*, then no non-zero vector* $\mathbf{x}$ *exists which satisfies* $A\mathbf{x} = \lambda\mathbf{x}$ *(ensuring the uniqueness of the solution* $\mathbf{x} = \mathbf{0}$*). Thus 1 is true. If* $\lambda = 0$*, and the solution* $\mathbf{x} = \mathbf{0}$ *is unique, then* A *must be invertible – but this was not given. If* $\lambda = 0$ *is an eigenvalue of* A*, then there exists a non-zero vector* $\mathbf{x} \neq 0$ *such that* $A\mathbf{x} = \mathbf{0}$ *which contradicts the uniqueness of the solution* $\mathbf{x} = \mathbf{0}$*. Consider* $A = I$ *and* $\lambda = 1$*, then* A *is invertible but any non-zero vector* $\mathbf{x}$ *satisfies* $A\mathbf{x} = \lambda\mathbf{x}$*. Thus invertibility is not necessary for the uniqueness of the solution* $\mathbf{x} = \mathbf{0}$*.*

Question 2

Let A be an $n \times n$ matrix with eigenvalue -1, I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Which of the following are true?

A. 0 is an eigenvalue of $A + I_n$.

B. $A + I_n$ is singular.

C. $A + I_n = 0_n$.

D. 1 is an eigenvalue of A^2.

Select from the following:

1. Only A, B and D.
2. Only A, B and C.
3. Only A, C and D.
4. All of A, B, C and D.
5. None of the above.

Answer: 1

Since -1 is an eigenvalue of A, there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} = -1\mathbf{x}$. *Since*

$$(A + I_n)\mathbf{x} = A\mathbf{x} + I_n\mathbf{x} = -\mathbf{x} + \mathbf{x} = \mathbf{0},$$

and $\mathbf{x} \neq \mathbf{0}$, *it follows that 0 is an eigenvalue of* $A + I_n$ *(A is true) and hence* $A + I_n$ *is singular (B is true). Let* $n = 2$ *and* $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. *It is straightforward to verify that A has -1 as an eigenvalue, but* $A + I_2 = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. *C is false. We have*

$$\begin{aligned} A^2\mathbf{x} &= A(A\mathbf{x}) = A(-1\mathbf{x}) = -A\mathbf{x} = -(-\mathbf{x}) \\ &= 1\mathbf{x} \end{aligned}$$

and since $\mathbf{x} \neq \mathbf{0}$, *D is true.*

Question 3

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A, C and D.
2. Only A.
3. Only A and B.
4. Only C and D.
5. None of the above.

Answer: 1

Since A is upper triangular, the eigenvalues (1, 2 and 3) lie on the main diagonal. Since the eigenvalues are all distinct, the matrix A is diagoalizable. The matrices C and D are both symmetric and therefore diagonalizable. The matrix B is upper triangular, with eigenvalues 0 (with algebraic multiplicity 1) and 1 (with algebraic multiplicity 2) which lie on the main diagonal. We find the eigenspace corresponding to the eigenvalue 1: the solutions of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

must satisfy $x + z = x$ and $0 = y$. It follows that $y = z = 0$ and the eigenspace is given by

$$\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}.$$

Thus the eigenspace corresponding to the eigenvalue 1 is the one-dimensional space

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The algebraic multiplicity (i.e. 2) and the geometric multiplicity (i.e. 1) of the eigenvalue 1 of B are not equal, the matrix B is not diagonalizable.

Question 4

Let A and B be $n \times n$ matrices and let I_n be the $n \times n$ identity matrix. Then

1. If ABB^TA^T is diagonalizable.
2. If A is diagonalizable then A is invertible.
3. If $\lambda = 0$ is an eigenvalue of A, then A is not diagonalizable.
4. If A and B are not diagonalizable then $A + B$ is not diagonalizable.
5. None of the above.

Answer: 1

We have $(ABB^TA^T)^T = (A^T)^T(B^T)^TB^TA^T = ABB^TA^T$. Thus ABB^TA^T is a symmetric matrix and consequently ABB^TA^T is diagonalizable. The matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is diagonalizable (distinct eigenvalue), but not invertible. The second implication is false. The matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

has $\lambda = 0$ as an eigenvalue and is diagonalizable. The third implication is false. The matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are not diagonalizable, but

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is symmetric and therefore diagonalizable. The fourth implication is false.

Question 5

Which one of the following defines an inner product?

1. $\langle p(x), q(x)\rangle = p(1)q(1) + 2p(2)q(2) + 3p(3)q(3)$ in P_2.
2. $\langle A, B\rangle = \text{tr}\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} AB^T\right)$ in M_{22}.
3. $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_2 + x_2y_1$ in $\mathbb{R}^2$.
4. $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + x_2y_2 - 1$ in $\mathbb{R}^2$.
5. None of the above.

Answer: 1

For option 1 we have

1. $\langle p(x), q(x)\rangle = p(1)q(1)+2p(2)q(2)+3p(3)q(3) = q(1)p(1)+2q(2)p(2)+3q(3)p(3) = \langle q(x), p(x)\rangle$.
2. $\langle (kp)(x), q(x)\rangle = (kp)(1)q(1) + 2(kp)(2)q(2) + 3(kp)(3)q(3)$
   $= kp(1)q(1)+2kp(2)q(2)+3kp(3)q(3) = k(p(1)q(1)+2p(2)q(2)+3p(3)q(3)) = \langle (x_1, x_2), (y_1, y_2)\rangle$.
3. $\langle p(x), (q + r)(x)\rangle = p(1)(q + r)(1) + 2p(2)(q + r)(2) + 3p(3)(q + r)(3)$
   $= p(1)(q(1) + r(1)) + 2p(2)(q(2) + r(2)) + 3p(3)(q(3) + r(3))$
   $= (p(1)q(1) + 2p(2)q(2) + 3p(3)q(3)) + (p(1)q(1) + 2p(2)q(2) + 3p(3)q(3))$
   $= \langle p(x), q(x)\rangle + \langle p(x), r(x)\rangle$
4. (a) $\langle p(x), p(x)\rangle = (p(1))^2 + 2(p(2))^2 + 3(p(3))^2 \geq 0$ *since* $(p(1))^2, (p(2))^2, (p(3))^2 \geq 0$.
   (b) $\langle p(x), p(x)\rangle = (p(1))^2 + 2(p(2))^2 + 3(p(3))^2 = 0$ *if and only if* $p(1) = p(2) = p(3) = 0$ *(again since* $x_1^2, x_2^2 \geq 0$*) if and only if* $p(x) = 0$ *(*$p(x)$ *is at most quadratic, i.e.* $p(x)$ *has at most 2 roots unless* $p(x) = 0$*).*

For option 2, consider $A = B = \left[\begin{smallmatrix}1 & 1\\ -1 & -1\end{smallmatrix}\right]$*. We have*

$$\left\langle \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right\rangle = \text{tr}\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}^T\right) = \text{tr}\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\right)$$
$$= \text{tr}\left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) = 0 + 0 = 0.$$

But $\left[\begin{smallmatrix}1 & 1\\ -1 & -1\end{smallmatrix}\right]$ *is not the zero vector in* M_{22}*. (Note that* $\begin{bmatrix}1 & -1\end{bmatrix}^T$ *is an eigenvector of* $\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]$ *corresponding to the eigenvalue 0.)*

For option 3 we have

$$\langle (1, 0), (1, 0)\rangle = 1 \cdot 0 + 0 \cdot 1 = 0$$

but $(1, 0)$ *is not the zero vector in* $\mathbb{R}^2$.

For option 4

$$\langle (1,0),(1,0)\rangle = 1\cdot 1 + 2\cdot 0\cdot 0 - 1 = 0$$

but $(1,0)$ *is not the zero vector in* $\mathbb{R}^2$. ***Alternative****: We could also have shown that*

$$\begin{aligned}\langle k(x_1,x_2),(y_1,y_2)\rangle &= \langle (kx_1,kx_2),(y_1,y_2)\rangle = kx_1y_1 + 2kx_2y_2 - 1\\ &\neq k(x_1y_1 + 2x_2y_2 - 1) = k\langle (x_1,x_2),(y_1,y_2)\rangle\end{aligned}$$

for $k \neq 1$. ***Alternative****: Lastly, we could have shown that*

$$\begin{aligned}\langle (x_1,x_2),(y_1,y_2)+(z_1,z_2)\rangle &= \langle (x_1,x_2),(y_1+z_1,y_2+z_2)\rangle = x_1(y_1+z_1) + 2x_2(y_2+z_2) - 1\\ &= x_1y_1 + 2x_2y_2 + x_1z_1 + 2x_2z_2 - 1\\ &\neq x_1y_1 + 2x_2y_2 - 1 + x_1z_1 + 2x_2z_2 - 1\\ &= \langle (x_1,x_2),(y_1,y_2)\rangle + \langle (x_1,x_2),(z_1,z_2)\rangle.\end{aligned}$$

Question 6

Which of the following vectors are unit vectors with respect to the inner product $\langle (x_1,x_2,x_3),(y_1,y_2.y_3)\rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3$ in $\mathbb{R}^3$?

A. $(1,0,0)$ B. $(0,1,0)/\sqrt{2}$ C. $(1,1,1)/\sqrt{3}$ D. $(1,1,0)/2$

Select from the following:

1. Only B and D.
2. Only A, C and D.
3. Only A and C.
4. Only A.
5. None of the above.

Answer: 1

$$\begin{aligned}\langle (1,0,0),(1,0,0)\rangle &= 2\cdot 1\cdot 1 + 2\cdot 0\cdot 0 + \cdot 0\cdot 0 = 2\\ &\neq 1 \textit{ (not a unit vector)}\\ \left\langle \left(0,\frac{1}{\sqrt{2}},0\right),\left(0,\frac{1}{\sqrt{2}},0\right)\right\rangle &= 2\cdot 0\cdot 0 + 2\cdot \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 0\cdot 0\\ &= 1 \textit{ (unit vector)}\\ \left\langle \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\rangle &= 2\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + 2\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = \frac{5}{3}\\ &\neq 1 \textit{ (not a unit vector)}\\ \left\langle \left(\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{2},\frac{1}{2},0\right)\right\rangle &= 2\cdot\frac{1}{2}\cdot\frac{1}{2} + 2\cdot\frac{1}{2}\cdot\frac{1}{2} + 0\cdot 0\\ &= 1 \textit{ (unit vector)}\end{aligned}$$

Question 7

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} A^T B\right)$ in M_{22} ?

A. $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. B. $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. C. $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$. D. $\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$.

Select from the following:

1. Only A and C are orthogonal, A and D are orthogonal, C and D are orthogonal.
2. Only B and C are orthogonal.
3. Only A and C are orthogonal, B and D are orthogonal.
4. Only A and C are orthogonal, A and D are orthogonal.
5. None of the above.

Answer: 4

We find

$$\left\langle \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}2 & 0\\ 4 & 0\end{bmatrix}\right) = 2$$

(not orthogonal)

$$\left\langle \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) = 0$$

(orthogonal)

$$\left\langle \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right) = 0$$

(orthogonal)

$$\left\langle \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}0 & 0\\ 4 & -4\end{bmatrix}\right) = -4$$

(not orthogonal)

$$\left\langle \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}0 & 0\\ 4 & 4\end{bmatrix}\right) = 4$$

(not orthogonal)

$$\left\langle \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}2 & 2\\ -4 & -4\end{bmatrix}\right) = -2$$

(not orthogonal)

Question 8

Consider the vector subspace $W = \text{span}\{\, 1 - x,\, 2x^2 \,\}$ of P_2 with the *standard inner product*. Which of the following vectors in P_2 lie in the subspace $W^{\perp}$ of P_2?

1. $x^2 + 1$.

2. $x + 1$.

3. $x - 1$.

4. $x^2 - 1$.

5. None of the above.

Answer: 2

If $p(x) \in W^{\perp}$ then for all $a, b \in \mathbb{R}$

$$\langle p(x), a(1-x) + b(2x^2)\rangle = a\langle p(x), 1-x\rangle + b\langle p(x), 2x^2\rangle = 0.$$

By choosing $a = 1$ and $b = 0$ we must have $\langle p(x), 1-x\rangle = 0$. By choosing $a = 0$ and $b = 1$ we must have $\langle p(x), x^2\rangle = 0$. Imposing these two conditions yields $\langle p(x), a(1-x) + bx^2\rangle = 0$. Thus it is sufficient to determine whether $\langle p(x), 1-x\rangle = 0$ and $\langle p(x), 2x^2\rangle = 0$. The standard inner products are given by

$$\begin{aligned}\langle a + bx + cx^2, 1 - x\rangle &= a\cdot 1 + b\cdot(-1) + c\cdot 0\\ &= a - b,\\ \langle a + bx + cx^2, 2x^2\rangle &= a\cdot 0 + b\cdot 0 + c\cdot 2\\ &= 2c.\end{aligned}$$

Thus we find

1. $\langle x^2 + 1, 1 - x\rangle = 1 - 0 = 1 \neq 0.$

2. $\langle x + 1, 1 - x\rangle = 1 - 1 = 0,$
   $\langle x + 1, 2x^2\rangle = 2\cdot 0 = 0.$

3. $\langle x - 1, 1 - x\rangle = -1 - 1 = 2 \neq 0.$

4. $\langle x^2 - 1, 1 - x\rangle = -1 - 0 = -1 \neq 0.$