Department of Decision Sciences

Introductory
Financial Mathematics

Only study guide for

DSC1630

University of South Africa
Pretoria

Cover: Eastern Transvaal, Lowveld (1928) J. H. Pierneef

J. H. Pierneef is one of South Africa's best known artists.
Permission for the use of this work was kindly granted
by the Schweickerdt family.

The tree structure is a recurring theme in various branches
of the decision sciences.

Contents

CHAPTER 1

Introduction

No business can exist without the information given by figures. One of the most important applications of information in business is as an aid for decision making.

In this guide we focus particularly on the use of financial information and we introduce a number of techniques used specifically to evaluate such information. Since borrowing, using and making money are at the heart (and soul!) of the commercial world, I need not emphasise the importance of information on financing. At the heart of most of these is the principle of interest and interest rate calculations. This leads into an examination of the principles involved in assessing the value of money over time and how this information can be used to evaluate alternative financial decisions.

We need to sound a word of caution before we start.

Remember that the financial decision area is a minefield in the real world, full of tax implications, depreciation allowances, investment and capital allowances and the like, which should be considered before making your final decisions.

This is one area in particular where the financial expert's help is essential. Nevertheless, the basic principles of such financial decision making are established through the concepts of interest and present value – two concepts we will discuss in detail in this guide.

So let's not waste time – let's start venturing into the interesting world of finance and specific interest calculations.

2

Simple interest and simple discount

Outcome of chapter

> To master the basic concepts and applications of simple interest and simple discount.

Key concepts

- ✓ Interest
- ✓ Simple interest
- ✓ Principal or present value
- ✓ Term
- ✓ Interest rate
- ✓ Amount or future value
- ✓ Due date or maturity date
- ✓ Time line
- ✓ Simple discount
- ✓ Discount rate
- ✓ Nominal or face value
- ✓ Number of days
- ✓ Time value of money

One of the characteristics that is said to distinguish adults from small children is that adults often delay compensation/rewards/payment – for instance, save half a bar of chocolate until tomorrow – whereas your average baby will almost certainly eat the lot now, even if it doesn't *really* want it. However there are circumstances, in which the baby's "grab it now" instinct is actually sounder than the adult's willingness to wait – though not for reasons that the baby could appreciate!

Most of the principles of financial decision making are based on a simple concept: that of *time preference*. All things being equal, we would prefer to receive a sum of money now rather than the same amount some time in the future. Offered a simple choice between R5 000 now and R5 000 in a year's time, the choice is clear: we would take the money now. Why? There are several reasons for this. What student doesn't need the spending money! But more importantly, there is an opportunity cost involved.

Consider your local bank. It approaches you, the customer, with the offer that if you deposit your savings of R5 000 with them for a year, they will, at the end of that time, return R5 000 to you. Such an offer is totally unattractive, I hear you say, given the "sacrifice" you would have to make over the next 12 months in going without the R5 000 for a year! You could invest it or you could use it to purchase goods and services from which you could derive satisfaction now. Clearly the bank has to offer

you some reward to part with your cash. In effect, the bank should offer to pay you not just R5 000 after a year, but R5 000 plus a financial reward. This financial reward is known as interest.

Interest rate calculations are straightforward and, because they form one of the basic building blocks of financial mathematics, we shall take some time to explore their principles.

2.1   Simple interest

Study unit 2.1

After working through this section you should be able to

> ▷ explain the terms

>> – interest and rate of interest

>> – principal and present value

>> – term (of a loan)

>> – sum accumulated and future value;

> ▷ relate the above-mentioned terms using the formula for simple interest and sum accumulated;

> ▷ calculate any one of the above-mentioned variables given the others;

> ▷ apply the simple interest formula to practical applications.

Let us define interest.

Definition 2.1 *Interest is the price paid for the use of borrowed money.*

Interest is paid by the party who uses or borrows the money to the party who lends the money. Interest is calculated as a fraction of the amount borrowed or saved (principal amount) over a certain period of time. The fraction, also known as the interest rate, is usually expressed as a percentage per year, but must be reduced to a decimal fraction for calculation purposes. For example, if we've borrowed an amount from the bank at an interest rate of 12% per year, we can express the interest as:

> 12% of the amount borrowed
> or 12/100 of the amount borrowed
> or 0,12× the amount borrowed.

When and how interest is calculated result in different types of interest. For example, simple interest is interest that is calculated on the principal amount that was borrowed or saved at the end of the completed term.

Now let's look at an example:

How much simple interest will be paid on a loan of R10 000 borrowed for a year at an interest rate of 10% per year?

10% of R10 000 must be paid as interest per year for the use of the money.

$$10\,000 \times 0{,}10 = 1\,000$$

The interest per year is R1 000.

Suppose we are using the money for two years.

$$10\,000 \times 0{,}10 \times 2 = 2\,000$$

Then the interest for the full loan period is R2 000.

In this case we have multiplied the amount used by the interest rate per year multiplied by the number of years' used.

Thus:

Definition 2.2 *Simple interest is interest that is computed on the principal for the entire term of the loan, and is therefore due at the end of the term. It is given by*

$$I = Prt$$

where

I *is the simple interest (in rand) paid at the end of the term for the use of the money*

P *is the principal or total amount borrowed (in rand) which is subject to interest (P is also known as the present value (PV) of the loan)*

r *is the rate of interest, that is, the fraction of the principal that must be paid each period (say, a year) for the use of the principal (also called the period interest rate)*

t *is the time in years, for which the principal is borrowed*

Note

1. The units used for the rate of interest and the term must be consistent. If, for example, the interest rate is calculated per annum, then the term must be in years, or a fraction thereof. If the interest rate is expressed for a shorter period (say, per month) then the term must be expressed accordingly (ie in months). It must be pointed out that, in this respect, a distinction is sometimes made between the so-called *ordinary interest year*, which is based on a 360-day year, and the *exact interest year*, which is based on a 365-day year. In the former case, each month has 30 days, each quarter 90 days, etc. In the latter, the exact number of days or months of the loan is used. Unless otherwise stated, we shall always work with exact periods.

2. In other textbooks you may find that different symbols are used for the different variables. For example, the formula for simple interest may be written as

$$I = Cin$$

   where C represents the capital or principal, i is the rate of interest and n is the number of periods. This should not be a source of confusion to you.

Definition 2.3 *The amount or sum accumulated* (S) *(also known as the maturity value, future value or accrued principal) at the end of the term* t, *is given by*

$$
\begin{aligned}
S &= \text{Principal value } + \text{ Interest} \\
S &= P + I \\
S &= P + Prt \\
S &= P(1 + rt).
\end{aligned}
$$

Note

When we replaced $S = P + Prt$ with $S = P(1 + rt)$ above, we applied the so-called "distributive law of multiplication over addition". This is a basic rule of mathematics that states

$$ab + ac = a(b + c).$$

In the above example we had $a = P$, $b = 1$ and $c = rt$.

Definition 2.4 *The date at the end of the term on which the debt is to be paid is known as the due date or maturity date.*

The following exercise will help to revise these concepts:

Exercise 2.1

> Calculate the simple interest and sum accumulated for R5 000 borrowed for 90 days at 15% per annum. See Appendix A p 149 of this study guide for the solution.

Sometimes we not only consider the basic formula $I = Prt$ but also turn it inside out and upside down, as it were, in order to obtain formulæ for each variable in terms of the others.

Of particular importance is the concept of *present value* P or PV, which is obtained from the basic formula for the sum or future value S, namely

$$S = P(1 + rt).$$

Dividing by the factor $(1 + rt)$ gives

$$P = \frac{S}{1 + rt}.$$

How do we interpret this result? We do this as follows: P is the amount that must be borrowed now to accrue to the sum S, after a term t, at interest rate r per year. As such it is known as the *present value* of the sum S. Stated formally:

Definition 2.5 *The present value of a debt* (S) *on a date prior to the due date is the value* $(P$ *or* $PV)$ *of the debt on the date in question and is given by the formula*

$$P = \frac{S}{1 + rt}$$

where r *is the interest rate and* t *is the "time to run to maturity".*

Again you should have no difficulty with the following exercise:

Exercise 2.2

A loan with a maturity value of R12 000 and an interest rate of 12% per annum
is paid three months prior to its due date. Determine the present value on the
day of the payment.

Study unit 2.2

2.2   Time lines

After working through this section you should be able to

▷ draw a time line to assist you in solving simple interest case problems relating
the future and present values via the interest received, and clearly indicating
the interest rate and term on it.

A useful way of representing interest rate calculations is with the aid of a so-called *time
line*. Time flow is represented by a horizontal line. Inflows of money are indicated by
an arrow from above pointing to the line, while outflows are indicated by a downward-
pointing arrow below the time line.

For a simple interest rate calculation, the time line is as follows:
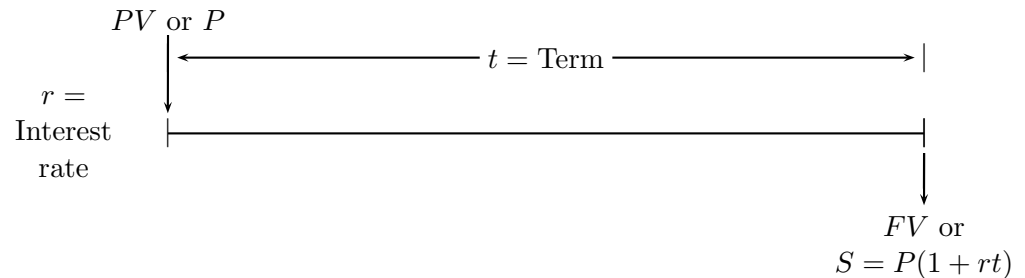
At the beginning of the term, the principal P (or present value) is deposited (or
borrowed) – that is, it is entered onto the line. At the end of the term, the amount or
sum accumulated, S (or future value) is received (or paid back). Note that the sum
accumulated includes the interest received. Remember that

NB

$$\text{Sum accumulated} \;=\; \text{Principal} \;+\; \text{Interest received}$$

that is

$$
\begin{aligned}
S &= P + Prt \\
  &= P(1 + rt)
\end{aligned}
$$

or equivalently

$$\text{Future value} \;=\; \text{Present value} \;+\; \text{Interest received}.$$

Using a time line, exercise 2.1 above may be represented as follows:
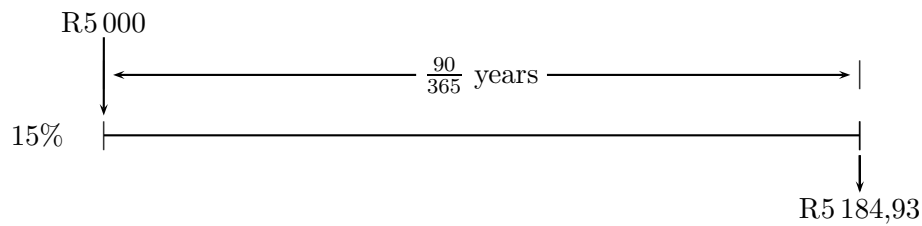
Similarly, exercise 2.2 above may be represented as follows:
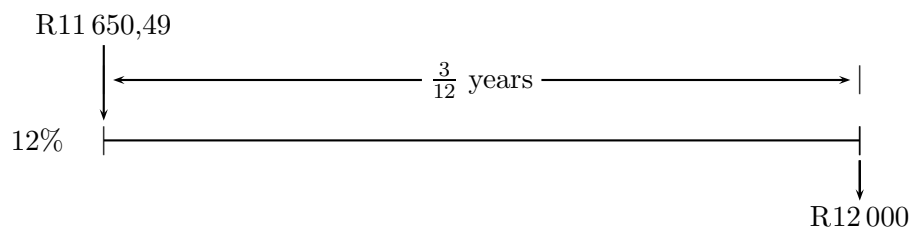
Note that the structure of these two problems is essentially the same. The only difference, apart from the figures, is that, in the former case, the future value was unknown, whereas in the latter it was the present value that we were asked to find.

2.3   Simple discount

Study unit 2.3

After working through this section you should be able to

▷ explain the terms

    − discount and discount rate

    − discounted or present value

    − face or future value

    − term;

▷ relate the above in a formula and represent it using a time line

▷ calculate

    − the simple discount for money market instruments

    − the effective interest rate for a given discount rate.

Definition 2.6 *Interest calculated on the face (future) value of a term and paid at the beginning of the term is called discount.*

In section 2.1 we emphasised the interest that has to be paid at the end of the term for which the loan (or investment) is made. On the due date, the principal borrowed *plus* the interest earned is paid back.

In practice, there is no reason why the interest cannot be paid at the beginning rather than at the end of the term. Indeed, this implies that the lender deducts the interest from the principal in advance. At the end of the term, only the principal is then due. Loans handled in this way are said to be *discounted* and the interest paid in advance is called the *discount*. The amount then advanced by the lender is termed the *discounted value*. The discounted value is simply the present value of the sum to be paid back and we could approach the calculations using the present value technique in section 2.1. Expressed in terms of the time line of the previous section, this means that we are given S and asked to calculate P (as was the case in exercise 2.2).
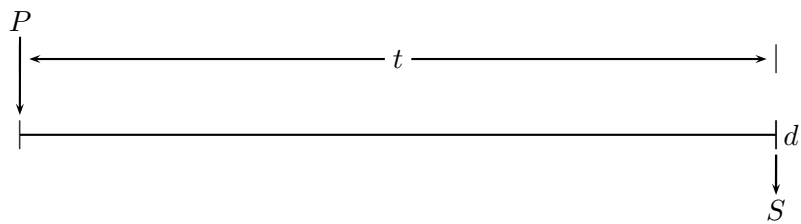
The discount on the sum S is then simply the difference between the future and present values. Thus the discount (D) is given by

$$D = S - P.$$

In the case of exercise 2.2, this would be

$$
\begin{aligned}
D &= 12\,000 - 11\,650{,}49 \\
  &= 349{,}51.
\end{aligned}
$$

Definition 2.7 *The discount D is given by*

$$D = Sdt$$

(compare to the formula for simple interest $I = Prt$) where d = simple discount rate and the discounted (or present) value of S is

$$
\begin{aligned}
P &= S - D \\
  &= S - Sdt \\
  &= S(1 - dt)
\end{aligned}
$$

or

Present Value = Future Value − Future Value × discount rate × time.

$$
\begin{aligned}
PV &= FV - FV \times d \times t \\
   &= FV(1 - dt)
\end{aligned}
$$

(compare to the formula for the accumulated sum or future value for simple interest $S = P(1 + rt)$).

This may be expressed in the form of the following time line:

PV = discounted or
present value

t = term of discount

$d =$
discount rate

FV = face value or
future value
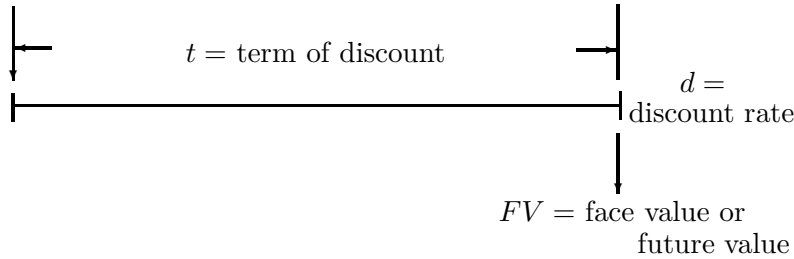
Two things are important here. Firstly, this is structurally similar to the time line for simple interest. Secondly, since we are working with *simple discount*, and the discount rate is now expressed as a *percentage of the future value* (and not as a percentage of the present value as before), a minus sign appears in the formula. This means that the discount, $FV \times d \times t$, is *subtracted* from the future value to obtain the present value.

Example 2.1

Determine the simple discount on a loan of R3 000 due in eight months at a discount rate of 15%. What is the discounted value of the loan? What is the equivalent simple interest rate r?

Now $S = 3\,000$, $d = 0{,}15$ and $t = \frac{8}{12} = \frac{2}{3}$.

This is represented by a time line

?

$t = \frac{8}{12}$

15%
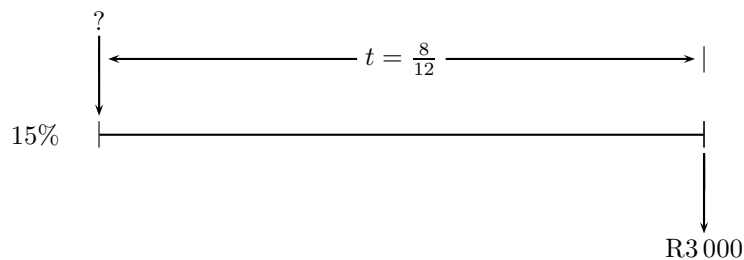
R3 000

Thus

$$
\begin{aligned}
D &= Sdt \\
&= 3\,000 \times 0{,}15 \times \frac{2}{3} \\
&= 300
\end{aligned}
$$

that is, the simple discount is R300.

$$
\begin{aligned}
P &= S - D \\
&= 3\,000 - 300 \\
&= 2\,700
\end{aligned}
$$

The discounted value is R2 700. In order to determine the equivalent interest rate r, we note that R2 700 is the price now and that R3 000 is paid back eight months later.

$$
\begin{aligned}
I &= S - P \\
&= 3\,000 - 2\,700 \\
&= 300
\end{aligned}
$$

The interest is thus R300. The question can thus be rephrased as follows: What simple interest rate, when applied to a principal of R2 700, will yield R300 interest in eight months?

But

$$I = Prt$$

and with substitute we get

$$300 = 2\,700 \times r \times \frac{2}{3}$$

that is

$$
\begin{aligned}
r &= \frac{3}{2} \times \frac{300}{2\,700} \\
&= 0{,}1667.
\end{aligned}
$$

Thus the equivalent simple interest rate is 16,67% per annum.

Note

Note the considerable difference between the interest rate of 16,67% and the discount rate of 15%. This emphasises the important fact that the interest rate and the discount rate are *not* the same thing. The point is that they act on different amounts, and at different times – the former acts on the present value, whereas the latter acts on the future value.

Exercise 2.3

1. A loan of R4 000 must be paid in six months time. Determine the amount of money that you will receive now if a simple discount rate of 18% is applicable. Determine the equivalent simple interest rate.

2. Determine the simple interest rate that is equivalent to a discount rate of

   (a) 12% for three months
   (b) 12% for nine months

   Hint: Let $S = 100$ and use the appropriate formulæ to set up an equation for r.

2.4   Counting days

After working through this section you should be able to

> ▷ determine the number of days between any two dates and apply this to problems involving simple interest and simple discount for a term specified by initial and final dates.

With simple interest and simple discount calculations it is often important to know the exact *number of days* between the beginning and end of the relevant term. For this purpose, the convention is as follows:

To calculate the exact number of days between the beginning and end of the relevant term, the day the money is lent (or deposited) is counted, but not the day the money is repaid (or withdrawn).

At its simplest, this means that if I deposit money in a bank today and withdraw it tomorrow I will receive interest for *one* day and not two. This is obviously the reason for counting in this way.

Many financial calculators have a "day function" which makes the calculation of the number of days between two dates quite easy. Alternatively, older financial textbooks often have tables from which the number sought can be read off with relative ease. For our purposes we will simply go ahead and "count on our fingers" (the original calculator!) OR use Appendix C p 201.

⌐**Example 2.2**

Determine the number of days between 19 March and 11 September of the same year.

List the months and relevant number of days in each month, and then determine the total.

Month	Number of days
March	13 (including 19 March*)
April	30
May	31
June	30
July	31
August	31
September	10 (excluding 11 September)
	$\underline{176}$

* Count on your fingers!

OR

Use Appendix C p 201 of this study guide.

Day number 254 (11 September) minus day number 78 (19 March) equals 176. ⌐

Exercise 2.4

Determine the number of days from 12 October to 15 May of the following year.

Exercise 2.5

Suppose an investor wishes to purchase a treasury bill (with a par value, that is face value, of R1 000 000,00) maturing on 2 July at a discount rate of 16,55% per annum and with a settlement date of 13 May of the same year. What would the required price be (this is present or discount value – also referred to as the consideration)? What is the equivalent simple interest rate of the investment?

Note

The settlement date is the date on which he, the investor, must pay.

Study unit 2.5

2.5 The time value of money

After working through this section you should be able to

▷ explain the concept of dated values of money and be able to illustrate this using time lines

▷ give rules for moving money forward or backward in time to a specific date

▷ apply the rules to obtain the value of a given debt (or investment) on different dates

▷ apply the time value concept to reschedule debts.

From what has been said before, it is evident that money has a *time value*. To be specific, we introduced the concepts "present value" and "future value" of a particular debt or investment, and showed that the two are related by the equation

$$\text{Future value} \; = \; \text{Present value} \; + \; \text{Interest}$$

or, in symbols,

$$\begin{aligned} S \; &= \; P + Prt \\ &= \; P(1 + rt). \end{aligned}$$

This means the following:

Definition 2.8 *A particular investment has different values on different dates.*

For example, R1 000 today will not be the same as R1 000 in six-months' time. In fact, if the prevailing simple interest rate is 16% per annum, then, in six months, the R1 000 will have accumulated to R1 080.

$$1\,000 \times (1 + 0{,}16 \times \frac{1}{2}) = 1\,080$$

On the other hand, three months ago it was worth less – to be precise, it was worth R961,54.

$$\frac{1\,000}{1 + 0{,}16 \times \frac{1}{4}} = 961{,}54$$

Represented on a time line, these statements yield the following picture:
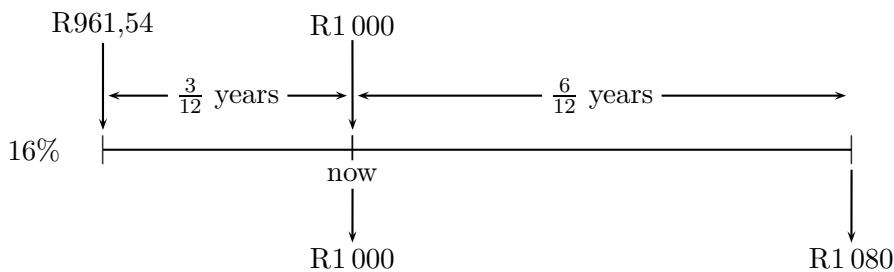
Note that this is actually two successive time line segments. In more general terms, we may depict the symbolic case as follows:
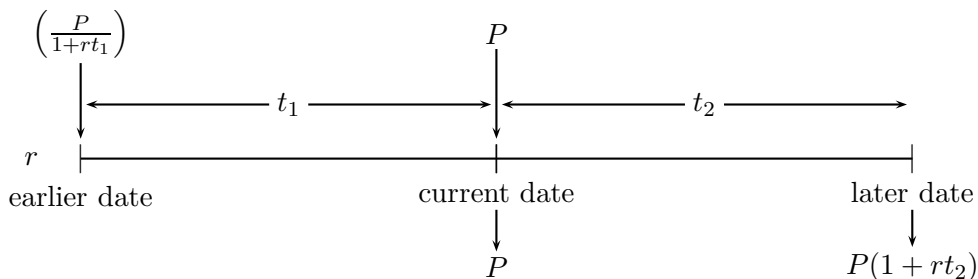
We can formulate the above results as two simple rules:

1. To move money *forward* (determine a future value) where simple interest is applicable, inflate the relevant sum by *multiplying* by the factor $(1 + rt)$.

2. To move money *backward* (determine a present value) where simple interest is applicable, deflate the relevant sum by *dividing* by the factor $(1 + rt)$.

Note: The second rule is equivalent to multiplying by a factor of $(1 + rt)^{-1}$.

NB

The point is that the mathematics of finance deals with dated values of money. This fact is fundamental to any financial transaction involving money due on different dates. In principle, every sum of money specified should have an attached date. Fortunately, in practice, it is often clear from the context what the implied date is.

⌐**Example 2.3**

Jack borrows a sum of money from a bank and, in terms of the agreement, must pay back R1 000 nine months from today. How much does he receive *now* if the agreed rate of simple interest is 12% per annum? How much does he owe after four months? Suppose he wants to repay his debt at the end of one year. How much will he have to pay then?

To answer the first question we must bring back the R1 000 nine months to today using the stated rate of simple interest.

This is represented on a time line.
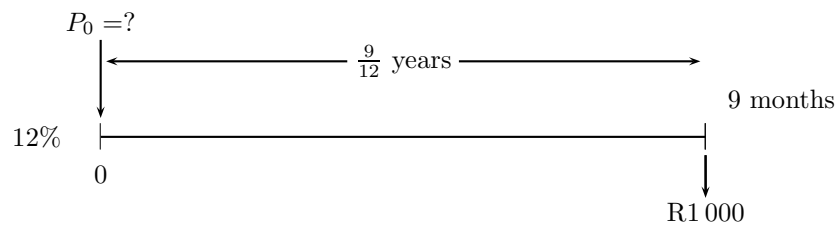
$$P_0 = \frac{1\,000}{1 + 0,12 \times \frac{9}{12}}$$
$$= 917,43$$

Thus the amount borrowed is R917,43. To determine how much he owes after four months we have to bring back the R1 000 by five months, as shown below on the time line.
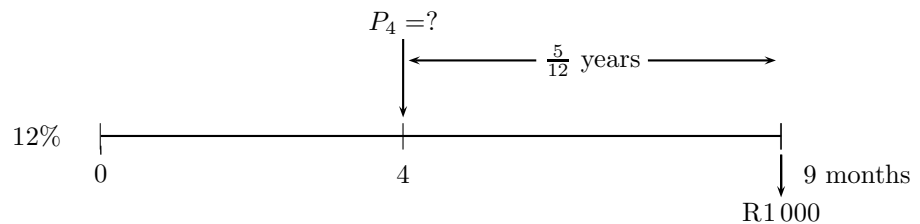
$$P_4 = \frac{1\,000}{1 + 0,12 \times \frac{5}{12}}$$
$$= 952,38$$

The present value at month four is R952,38. Finally, to calculate the amount owed at the end of a year, we have to move the R1 000 forward by three months.

This is represented on a time line.

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$$P_{12} \quad = \quad 1\,000 \times (1 + 0{,}12 \times \frac{3}{12})$$
$$= \quad 1\,030$$

The amount owed at the end of the year is R1 030,00. Notice how I have used a subscript attached to the letter P to indicate the relevant month to which the money is referring to.

Exercise 2.6

Melanie owes R500 due in eight months. For each of the following cases, what single payment will repay her debt if money is worth 15% simple interest per annum?

(a) now

(b) six months from now

(c) in one year

From time to time a debtor may wish to replace a set of financial obligations with a single payment on a given date. In fact, this is one of the most important problems in financial mathematics. It must be emphasised here that the sum of a set of dated values due on different dates has *no meaning*. *All dated values must first be transformed to values due on the same date* (normally the date on which the payment that we want to calculate is due). The process is simply one of repeated application of the above two key steps as the following example illustrates:

Example 2.4

Maxwell Moneyless owes Bernard Broker R5 000 due in three months and R2 000 due in six months. Maxwell offers to pay R3 000 immediately if he can pay the balance one year from now. Bernard agrees, on condition that they use a simple interest rate of 16% per annum. They also agree that for settlement purposes the R3 000 paid now will also be subject to the same rate. How much will Maxwell have to pay at the end year one? (Take the comparison date as one year from now!)

All moneys are shown in the following time diagram with debts above the line and payments below:
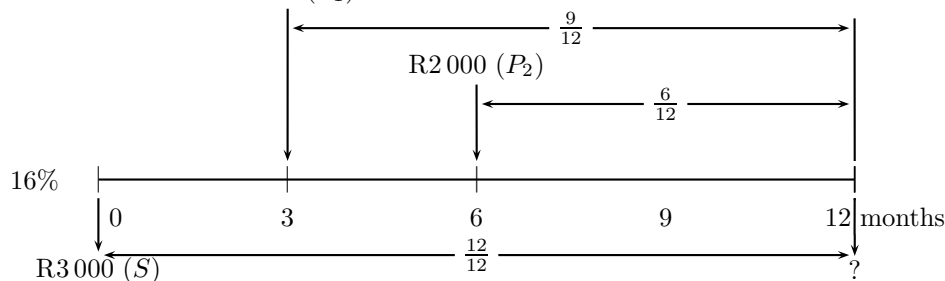
The values of the R5 000 and R2 000 debts in 12 months are:

$$P_1 \quad = \quad 5\,000 \times \left(1 + 0{,}16 \times \frac{9}{12}\right)$$
$$= \quad 5\,600$$

and

$$P_2 = 2\,000 \times \left(1 + 0{,}16 \times \frac{6}{12}\right)$$
$$= 2\,160.$$

The value of the total debt at year end R7 760 (5 600 + 2 160).

The value of the R3 000 (S) repayment in 12 months is

$$S = 3\,000 \times (1 + 0{,}16 \times 1)$$
$$= 3\,480.$$

The outstanding debt to be paid in 12 months is R4 280 (7 760 − 3 480).

Note

If we took the date "now" as the date to transform values to, we would get R4 245,07.

This is represented on a time line.
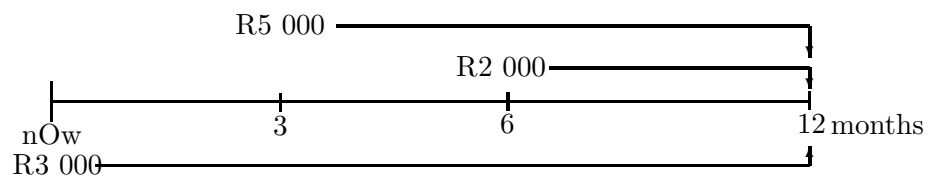
If we move the debts (the R5 000 and R2 000) back to now and then forward this amount to month 12 we will get:

$$\text{Debt at now} = 5000 \left(1 + 0{,}16 \times \tfrac{3}{12}\right)^{-1} + 2000 \left(1 + 0{,}16 \times \tfrac{6}{12}\right)^{-1}$$
$$= 4\,807{,}69 + 1\,851{,}85$$
$$= 6\,659{,}54$$

This amount of R6 659,54 must now be moved forward to month 12.

$$\text{Debt} = 6659{,}54 \left(1 + 0{,}16 \times \tfrac{12}{12}\right)$$
$$= 7\,725{,}07$$

The total debt is R7 725,07.

The payment of R3 000 must be moved forward to month 12.

$$\text{Payment} = 3000 \left(1 + 0{,}16 \times \tfrac{12}{12}\right)$$
$$= 3\,480{,}00$$

The outstanding debt to be paid at month 12 is R4 245,07 (7 725,07 − 3 480).

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That there should indeed be a difference in the two answers becomes clear when we consider the maths involved:

Suppose an amount of R100 is due in n months ($1 \leq n \leq 11$). Now we defer payment of the amount to the end of the year, but let interest accrue at a rate of $r\%$ per annum over this period.

At the end of the year the following payment will then be due

$$100 \times \left(1 + r \times \frac{12 - n}{12}\right).$$

For example:

Let $n = 5$ and $r = 16\%$ then

$$
\begin{aligned}
& 100 \left(1 + r \times \frac{12 - n}{12}\right) \\
= \ & 100 \left(1 + 0{,}16 \times \frac{12 - 5}{12}\right) \\
= \ & 100 \left(1 + 0{,}16 \times \frac{7}{12}\right) \\
= \ & 109{,}33.
\end{aligned}
$$

The amount due at the end of the year is R109,33. Will we get the same result if we first move the R100 debt back to the beginning of the year, and then move it forward again to the end of the year? Let's see.

First move the amount back n months. This gives:

$$\frac{100}{1 + r \times \frac{n}{12}}.$$

Now we move it one year forward and get:

$$\frac{100}{1 + r \times \frac{n}{12}}(1 + r) = \frac{100(1 + r)}{1 + r \times \frac{n}{12}}.$$

Again we let $n = 5$ and $r = 16\%$. Then

$$
\begin{aligned}
& \left(\frac{100(1 + r)}{1 + r \times \frac{n}{12}}\right) \\
= \ & \frac{100(1 + 0{,}16)}{1 + 0{,}16 \times \frac{5}{12}} \\
= \ & 108{,}75.
\end{aligned}
$$

The amount due at the end of the year is R108,75. Equality does not hold, which explains why we get different results when the debts are moved forward to the settlement date, and when the debts are moved backward to "now" and the resulting debt is then moved forward to the end of the year.

The reason for this apparent anomaly is that simple interest is, by definition, a specific rate for a specific time. When we apply the rate to a different period than the one for which is was specified, it actually does not make sense. In the example, the intention is that the 16% interest on the debts of R5 000 and R3 000 accrues to the end of the year, and therefore the correct answer is obtained when the debts are moved forward.

Exercise 2.7

Mary-Anne must pay the bank R2 000 which is due in one year. She is anxious to lessen her burden in advance and therefore pays R600 after three months, and another R800 four months later. If the bank agrees that both payments are subject to the same simple interest rate as the loan, namely 14% per annum, how much will she have to pay at the end of the year to settle her outstanding debt?

2.6   Summary of Chapter 2

In this chapter the basic concept and formulæ pertaining to *simple interest* were discussed. The basic formula for simple interest is

$$S = P(1 + rt)$$

where P is the principal invested for a term t at rate r per year, and S the sum accumulated at simple interest.

The *present value*, P, of S is defined as

$$P = \frac{S}{1 + rt}.$$

The concept of *a time line* that relates the present and future values graphically was introduced and shown to be a useful tool:
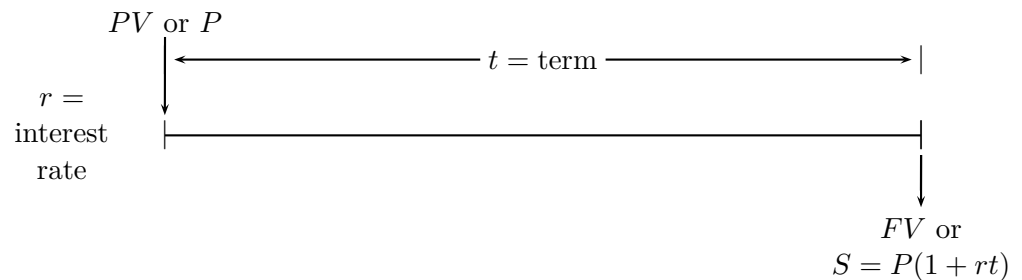
The inverse concept of *simple discount* was discussed. The discounted (or present) value of S was defined as

$$
\begin{aligned}
P &= S - D \\
&= S(1 - dt)
\end{aligned}
$$

where d is the discount rate.

Finally, the above concepts of present and future value were used to introduce the *time value* of money. It was stressed that in principle the value of any debt or investment must be linked to a specific date.

The relationship between values on different dates is summarised by the following time line:
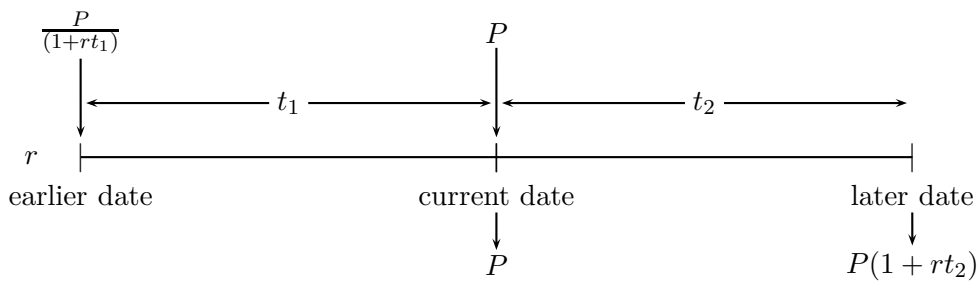
2.7   Evaluation exercises

See Appendix B p179 of this guide for the solutions.

1. At what rate of simple interest will R600 amount to R654 in nine months?

2. A loan was issued on 1 April for R1 500, and is payable on 1 October of the same year. A simple interest rate of 16% per annum, is applicable. What is its present value if the loan was paid on 1 July of the same year?

3. The simple discount rate of a bank is 16% per annum. If a client signs a note to pay R6 000 in nine months time, how much will the client receive? What is the equivalent simple interest rate?

4. Anthony borrowed R3 000 on 4 February at a simple interest rate of 15% per annum. He paid R1 000 on 21 April of the same year, R600 on 12 May of the same year, and R700 on 11 June of the same year. What was the balance due on 15 August of the same year if payments were subject to the same simple interest as the original debt?

Compound interest and equations of value

Outcome of chapter

> To master the basic concepts and applications of compound interest and other interest rate formulæ.

Key concepts

✓ Compound interest	✓ Odd period
✓ Compounded amount	✓ Fractional compounding
✓ Present value	✓ Continuous compounding
✓ Nominal rate	✓ Equation of value
✓ Effective interest rate	✓ Comparison date

Study unit 3.1

3.1   Compound interest

> *After working through this section you should be able to*
>
> ▷ explain the relationship between compounded amount, principal, interest rate and term, and be able to give the compound interest formula which relates these entities, as well as be able to depict this relationship using a time line;
>
> ▷ calculate any one of the above-mentioned variables given the others;
>
> ▷ apply the formulæ to practical applications.

Compound interest arises when, in a transaction over an extended period of time, interest due at the end of a payment period is not paid, but added to the principal. Thereafter, the interest also earns interest, that is, it is compounded. The amount due at the end of the transaction period is the *compounded amount* or *accrued principal* or *future value*, and the difference between the compounded amount and the original principal is the *compound interest*. Essentially, the basic idea is that interest is earned on interest previously earned. For example, if you have a savings account and, instead

of withdrawing the interest earned at the end of the year, you leave it in the account to be added to the principal, then at the end of the next year you will receive interest on both the original principle and your first year's interest. We say that your interest has *compounded*. Leave it in for another year and it will be further compounded. To be specific consider the following example:

Example 3.1

You deposit R600 in a savings account that offers you interest of 10% per annum calculated annually. If you do not withdraw any of the interest, but allow it to be added to the principal at the end of each year, how much would you have after three years?

At the end of the first year

$$\begin{aligned} I_1 &= 600 \times 0{,}1 \times 1 \\ &= 60. \end{aligned}$$

The interest earned is R60. This R60 is now added to the principal, which becomes R660.

The interest earned in the second year is

$$\begin{aligned} I_2 &= 660 \times 0{,}1 \times 1 \\ &= 66. \end{aligned}$$

The principal available for the third year is thus R726 $(660 + 66)$.

Finally the interest for the third year is

$$\begin{aligned} I_3 &= 726 \times 0{,}1 \times 1 \\ &= 72{,}60. \end{aligned}$$

Thus after three years, your R600 investment will have grown to R798,60 $(726 + 72{,}60)$ and the compound interest earned will be R198,60 $(798{,}60 - 600)$.

Note

1. For each year, the calculation is in fact a simple interest calculation with the term $t = 1$.

2. Note the use of the subscript 1, 2, or 3 attached to the interest symbol I to identify the year in each case.

As I have shown in the example, compound interest is in fact just the repeated application of simple interest to an amount that is at each stage increased by the simple interest earned in the previous period. It is, however, obvious that where the investment term involved stretches over many periods, compound interest calculations along the above lines can become tedious. Is there perhaps a formula for calculating the amount generated for any number of periods? There is, and in fact we can derive it for ourselves.

We start with our basic formula for simple interest:

$$I = Prt$$

which we must apply repeatedly as we did in the last example. Remember that Prt stands for $P \times r \times t$, that is, we suppress the multiplication signs. However, since we work with one period at a time, we may set $t = 1$. Starting with a basic principal P and at an interest rate of r, the interest earned in the first period is

$$I_1 = Pr.$$

Our basic principal invested for the second period (P_2) is the sum of the original principal and the interest earned, that is,

$$
\begin{aligned}
P_2 &= P + I_1 \\
&= P + Pr \\
&= P(1 + r).
\end{aligned}
$$

We use the distributive law of multiplication over addition in the last step. Note the use of the symbol P_2 (that is P subscript 2) to indicate the basic principal for the second period.

The interest earned in the second period is then

$$
\begin{aligned}
I_2 &= P_2 r \\
&= P(1 + r)r
\end{aligned}
$$

(that is the simple interest formula applied with $P(1 + r)$ instead of P, and $t = 1$).

Our basic principal for the third period (P_3) is then

$$
\begin{aligned}
P_3 &= P_2 + I_2 \\
&= P(1 + r) + P(1 + r)r \\
&= P(1 + r)(1 + r) \\
&= P(1 + r)^2.
\end{aligned}
$$

Again, we applied the distributive law, taking $P(1 + r)$ as a single number for this purpose in the second step.

In the last step we used $(1+r)(1+r) = (1+r)^2$, which is the definition of the square of a number.

The interest earned in the third period is then

$$
\begin{aligned}
I_3 &= P_3 r \\
&= P(1 + r)^2 r
\end{aligned}
$$

and the principal available at the end of the third period for investment in the fourth (P_4) is

$$
\begin{aligned}
P_4 &= P_3 + I_3 \\
&= P(1 + r)^2 + P(1 + r)^2 r \\
&= P(1 + r)^2(1 + r)
\end{aligned}
$$

again using the distributive law.

But, according to the law of exponents,

$$(1+r)^2(1+r) = (1+r)^3.$$

Thus the principal available after the third period for investment in the fourth period is

$$P_4 = P(1+r)^3.$$

By again using the same formula and following the same steps we find that the principal available after the fourth period is

$$P_5 = P(1+r)^4. \quad \text{(Check for yourself!)}$$

We could continue in this way but I'm sure that the formula is already evident, namely that after the nth period, the accrued principal, or compounded amount (S), will be

$$S = P(1+r)^n.$$

The accrued compound interest is simply the difference between S and the original principal invested, that is, $S - P$.

Once again I emphasise that the symbols used are arbitrary. In many textbooks you will find other symbols.

I suggest that we use n for the number of periods (that is terms).

The formula for compound interest is thus as follows:

Definition 3.1

$$S = P(1+i)^n$$

where S (also known as the future value) is the amount which accrues when an initial principal of P is invested at interest rate i per period for a term of n periods. The compound interest, I, is

$$I = S - P.$$

The present value is P, and S is the future value.

Note

For compound interest, we will use the symbol i for the interest rate and n for the period.

In terms of a time line, compound interest may be represented as follows:
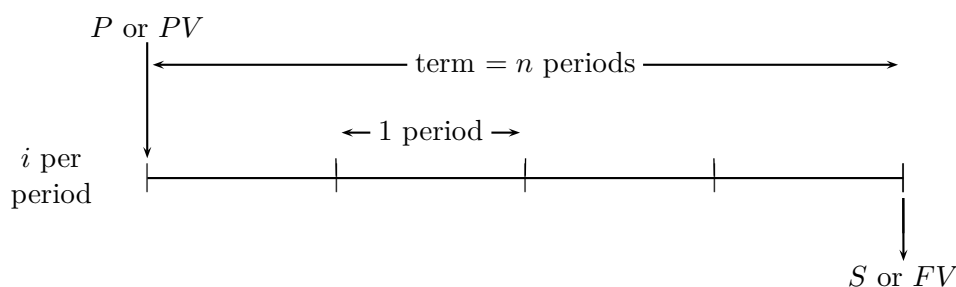
Let's see how our formula works. Refer back to our last example where we used $P = 600$, $i = 0,1$ and $n = 3$. Now

$$
\begin{aligned}
S &= P(1 + i)^n \\
S &= 600(1 + 0,1)^3 \\
&= 600(1,1)^3 \\
&= 798,60.
\end{aligned}
$$

We therefore obtain $S = \text{R}798,60$ as before.

Of course, you could also have used your calculator to obtain this result in one step.

Exercise 3.1

1. Find the compounded amount on R5 000 invested for ten years at $7\frac{1}{2}\%$ per annum compounded annually.

2. How much interest is earned on R9 000 invested for five years at 8% per annum and compounded annually?

Perhaps you have noticed that I have been careful to use the phrase "compounded annually" in the above examples and exercises. This is because the compound interest earned depends a lot on the intervals or periods over which it is compounded. Financial institutions frequently advertise investment possibilities in which interest is calculated at intervals of less than a year, such as semi-annually, quarterly, monthly or even on "daily balance". What difference does this make? A few examples should help us answer this question.

Example 3.2

R1 000 is invested for one year at 8% per annum but the interest is compounded semi-annually (ie every six months). How much interest is earned? Compare the interest earned to the simple interest which would be earned at the same rate in one year.

The interest earned after six months is obtained using the simple interest formula with $t = \frac{1}{2}$ (that is $6 \div 12$).

Thus

$$
\begin{aligned}
I_1 &= P \times r \times t \\
&= 1\,000 \times 0,08 \times \frac{1}{2} \\
&= 40.
\end{aligned}
$$

The interest earned is R40. For the second half of the year the principal is R1 040 ($1\,000 + 40$).

$$
\begin{aligned}
I_2 &= 1\,040 \times 0,08 \times \frac{1}{2} \\
&= 41,60
\end{aligned}
$$

The interest earned is R41,60. The total compound interest earned during the year is R81,60 ($40 + 41,60$). If simple interest is applicable:

$$
\begin{aligned}
I &= 1\,000 \times 0,08 \times 1 \\
&= 80
\end{aligned}
$$

25

The interest earned during the year would be R80. Thus, for the same rate the compound interest earned is more than the simple interest. ⌟

Note

Instead of working out the interest for the above example, step by step, we could have applied our compound interest formula. How? Think back for a moment and you will realise that when deriving the formula we did not actually speak of years – only of the number of periods, n, and the interest rate per period, i. In this example the number of periods per year is two, while the interest rate per period is $\frac{1}{2} \times 8\%$ that is 4%, since interest is calculated twice, at the middle of the year and at the end.

Thus, applying $S = P(1 + i)^n$ with $P = 1\,000$, $i = 0,04$ and $n = 2$ we find that $S = 1\,081,60$ so that the interest earned is R81,60. (Check for yourself!)

Example 3.3

Suppose that in example 3.2 the interest is compounded quarterly. How much will be earned?

The number of periods per year is 4, and the interest rate per period is $\frac{1}{4} \times 8\%$, that is, 2%.

Thus $P = 1\,000$, $i = 0,02$ and $n = 4$ and the compound interest formula yields $S = R1\,082,43$ so that the interest earned is R82,43. (Check for yourself again.) ⌟

In other words, our compound interest formula is applicable as it stands. *We must just remember that the interest rate for each period is determined by dividing the annual interest rate by the number of compounding periods per year, while the total number of compounding periods is the number of compounding periods per year times the number of years' for which the investment is made.*

To make it clearer, however, we will redefine the formula for compound interest introducing some new symbols. It is still the same formula, but we now differentiate clearly between the annual interest rate, j_m, and the interest rate per compounding period, i, as well as the term, t, in years, and n the number of compounding periods.

The new formula for compound interest is now as follows:

$$S = P(1 + i)^n$$

NB

where
S	≡	the accrued amount, also known as the future value
P	≡	the initial principal, also known as the present value
i	≡	$\frac{j_m}{m}$, the annual interest rate compounded m times per year
n	≡	$t \times m$, = number of compounding periods
t	≡	the number of years' of investment
m	≡	the number of compounding periods per year
j_m	≡	the nominal interest rate per year

Therefore
$$S = P(1 + i)^n$$

or
$$S = P\left(1 + \frac{j_m}{m}\right)^{tm}$$

or
$$S = P\left(1 + \frac{j_m}{m}\right)^{n}$$

or
$$S = P(1 + i)^{tm}.$$

Exercise 3.2

R2 000 is invested at 14% per annum for three years. Determine the accrued principal if interest is calculated (and added to the principal)

(a) yearly

(b) half-yearly

(c) quarterly

(d) daily.

Draw the time line in each case.

A question we often have to answer regarding investments is the following: How much must be invested now to generate a given amount at the end of a specified term? This can be answered by simply rearranging our basic formula for compound interest.

Definition 3.2 *As was the case for simple interest, we may define the present value (P) as the amount that must be invested (or borrowed) for n interest periods, at a rate of i per interest period, in order to accumulate the amount S on the due date. We obtain this by solving for P from the compound interest formula as follows:*

If
$$S = P(1 + i)^n \qquad : \qquad S = P\left(1 + \frac{j_m}{m}\right)^{tm}$$

then
$$P = \frac{S}{(1 + i)^n} \qquad : \qquad P = \frac{S}{\left(1 + \frac{j_m}{m}\right)^{tm}}$$

or
$$P = S(1 + i)^{-n} \qquad : \qquad P = S\left(1 + \frac{j_m}{m}\right)^{-tm}.$$

Note

$(1 + i)^{-n}$ is mathematical shorthand for $\frac{1}{(1+i)^n}$.

Exercise 3.3

What is the present value if the compounded amount (future value) R10 000 is invested at 18% per annum for five years and interest is calculated monthly?

We know that $2 \times 2 \times 2 = 8$ and that it can be written as

$$2^3 = 8$$

the so-called "exponential form".

This same equation can be written in logarithmic form as

$$\log_2 8 = 3.$$

Now any positive number x can be written as

$$x = e^y$$

and

$$\log_e x = y$$

and since log to the base e can be written as ln we can say that (ie $\log_e = \ln$)

$$\ln x = y.$$

The ln (in words "lin") key on your calculator means the logarithm to base e.

Note

The number e is defined as the value of the equation

$$\left(1 + \frac{1}{m}\right)^m$$

where m is a number that is so large it approaches infinity.

The exponential function e^x and the logarithmic function $\ln y$ are related in that if

$$e^x = y$$
$$x \ln e = \ln y$$

then

$$\ln y = x. \qquad (\ln e = 1)$$

For example, given $x = 2{,}30259$

then

$$e^{2{,}30259} \approx 10$$

consequently $\ln 10 \approx 2{,}30259$. (Verify it on your calculator.)

Definition 3.3 *The natural logarithm,* $\ln$*, (pronounced "lin") is defined as follows:*

$$y = \ln x \ \text{if and only if} \ x = e^y \ \text{and} \ x > 0.$$

Also, according to the logarithmic laws,

$$\ln m^n = n \ln m.$$

If a and b are equal, then their natural logarithm will also be equal, so if $a = b$ then $\ln a = \ln b$.

We have defined the accumulated value S of an initial principal P invested at an interest rate i per period for a term of n periods as

$$S = P(1 + i)^n.$$

In order to calculate n we must manipulate the formula as follows:

$$
\begin{aligned}
S &= P(1 + i)^n \\
(1 + i)^n &= \frac{S}{P} \qquad &\text{(get rid of the coefficient } P) \\
n \ln(1 + i) &= \ln\left(\frac{S}{P}\right) \qquad &\text{(take ln both sides)} \\
n &= \frac{\ln\left(\frac{S}{P}\right)}{\ln(1 + i)}. \qquad &\text{(get rid of } \ln(1 + i))
\end{aligned}
$$

According to the formula

$$
\begin{aligned}
S &= P\left(1 + \frac{j_m}{m}\right)^{tm} \\
tm &= \frac{\ln\left(\frac{S}{P}\right)}{\ln\left(1 + \frac{j_m}{m}\right)} \\
t &= \frac{\ln\left(\frac{S}{P}\right)}{m \ln\left(1 + \frac{j_m}{m}\right)}.
\end{aligned}
$$

Exercise 3.4

   After what period of time will R2 500 double if interest is calculated half-yearly at 11% per annum?

To obtain i and j_m if P, S, n and m are given, we work as follows:

Start again with the equation

$$S = P(1+i)^n.$$

Then $$(1+i)^n = \left(\frac{S}{P}\right)$$

We find that $$1+i = \left(\frac{S}{P}\right)^{\frac{1}{n}} \quad \text{(get rid of the power } n\text{)}$$

and $$i = \left(\frac{S}{P}\right)^{\frac{1}{n}} - 1.$$

But $$i = \frac{j_m}{m} \quad \text{and} \quad n = tm.$$

Then $$\frac{j_m}{m} = \left(\frac{S}{P}\right)^{\frac{1}{tm}} - 1$$

and $$j_m = m\left(\left(\frac{S}{P}\right)^{\frac{1}{tm}} - 1\right).$$

This may be daunting but your calculator won't blink twice.

Exercise 3.5

At what interest rate per annum must money be invested if the accrued principal must treble in ten years?

3.2   Nominal and effective interest rates

After working through this section you should be able to

▷ explain the concept of effective rate of interest and how it depends on both the compounding period and the given nominal rate;

▷ calculate the effective rate of interest given the nominal rate and the compounding period;

▷ convert a given nominal rate to another nominal rate.

Another vital concept in interest calculations is that of *nominal* versus *effective rate* of interest.

Definition 3.4 *In cases where interest is calculated more than once a year, the annual rate quoted is the nominal annual rate or nominal rate.*

For example, in exercise 3.2, the rate of 14% per annum is the *nominal rate* in all four cases. However:

Definition 3.5 *If the actual interest earned per year is calculated and expressed as a percentage of the relevant principal, then the so-called "effective interest rate" is obtained. This is the equivalent annual rate of interest – that is, the rate of interest actually earned in one year if compounding is done on a yearly basis.*

An example should clarify this concept.

Example 3.4

Calculate the effective rates of interest if the nominal rate is 15% per annum
and interest is calculated

(a) yearly

(b) half-yearly

(c) quarterly

(d) monthly

(e) daily

(a) If interest is calculated yearly, then there is, by definition, no difference between
the nominal and effective rates, which are both 15% in this case.

(b) We could use any principal, but it is convenient to choose $P = $ R100.
For half-yearly compounding, $j_m = 0{,}15$, $m = 2$ and $t = 1$.

$$
\begin{aligned}
S &= P\left(1+i\right)^n \\
&= P\left(1+\frac{j_m}{m}\right)^{tm} \\
&= 100\left(1+\frac{0{,}15}{2}\right)^2 \\
&= 115{,}56
\end{aligned}
$$

The interest is R15,56 $(115{,}56-100)$ and expressed as a percentage, this is 15,56%
per annum (since $P = $ R100), which is the effective rate.

(c) For quarterly compounding, $j_m = 0{,}15$, $m = 4$ and $t = 1$.

$$
\begin{aligned}
S &= 100\left(1+\frac{0{,}15}{4}\right)^4 \\
&= 115{,}87
\end{aligned}
$$

The interest is R15,87 $(115{,}87 - 100)$ and the effective rate is 15,87% per annum.

(d) For monthly compounding, $j_m = 0{,}15$, $m = 12$ and $t = 1$.

$$
\begin{aligned}
S &= 100\left(1+\frac{0{,}15}{12}\right)^{12} \\
&= 116{,}08
\end{aligned}
$$

The interest is R16,08 $(116{,}08 - 100)$ and the effective rate is 16,08% per annum.

(e) For daily compounding, $j_m = 0{,}15$, $m = 365$ and $t = 1$.

$$
\begin{aligned}
S &= 100\left(1+\frac{0{,}15}{365}\right)^{365} \\
&= 116{,}18
\end{aligned}
$$

The interest is R16,18$(116{,}18 - 100)$ and the effective rate is 16,18% per annum.
We can therefore conclude that the effective rate increases as the number of
compounding periods increases.

From the above example you should note that, in order to calculate the effective rate, we do not require the actual principal involved. In fact, it is convenient to use $P = 100$, since the interest calculated then immediately yields the effective rate as a percentage. This can then be used to formulate an equation for the effective rate. Suppose that the number of times the interest is calculated each year is m and that the nominal rate is j_m. Thus $i = \frac{j_m}{m}$. The amount S accumulated in a year if $P = 100$ will equal

$$
\begin{aligned}
S & = P\left(1 + \frac{j_m}{m}\right)^{tm} \\
& = 100\left(1 + \frac{j_m}{m}\right)^{tm} \qquad \text{(but } t = 1) \\
& = 100\left(1 + \frac{j_m}{m}\right)^{m}.
\end{aligned}
$$

(Compare this with the above example in which $n = 1,\ 2,\ 4,\ 12$ and 365 for the five cases respectively.)

The interest rate (expressed as a percentage) is then

$$
\begin{aligned}
J_{eff} & = S - 100 \\
& = 100\left(1 + \frac{j_m}{m}\right)^{m} - 100 \\
& = 100\left(\left(1 + \frac{j_m}{m}\right)^{m} - 1\right).
\end{aligned}
$$

The value J_{eff} calculated in this way is called the *effective interest rate* expressed as a percentage. (Again, compare this with the example.)

The effective interest rate expressed as a percentage is given by

$$
J_{eff} = 100\left(\left(1 + \frac{j_m}{m}\right)^{m} - 1\right).
$$

NB

The above result may be summarised in terms of a time line as follows:
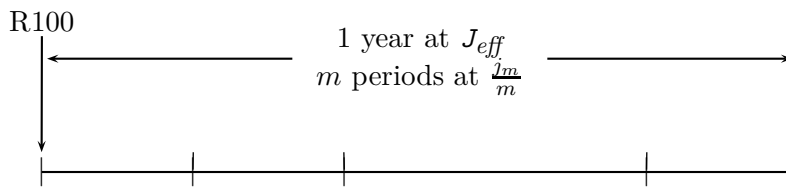
$$
\begin{aligned}
S & = 100\left(1 + \frac{J_{eff}}{100}\right) \\
& = 100 + J_{eff} \\
& = 100 + 100\left(\left(1 + \frac{j_m}{m}\right)^{m} - 1\right) \\
& = 100\left[1 + \left(1 + \frac{j_m}{m}\right)^{m} - 1\right] \\
& = 100\left(1 + \frac{j_m}{m}\right)^{m}
\end{aligned}
$$

Exercise 3.6

Use the formula to calculate the effective interest rate J_{eff} (expressed as a percentage) if the nominal rate is 22% and the interest is calculated

(a) half-yearly

(b) quarterly

(c) monthly

(d) daily.

The effective rate corresponding to any nominal rate does not depend on the amount invested. Suppose you invest P rand at the beginning of the year for one year at a nominal interest rate of j_m per year, compounded m times per year.

The accumulated amount that you will receive at the end of the year is

$$S = P\left(1 + \frac{j_m}{m}\right)^m. \qquad (t = 1)$$

The effective rate for the investment for that year is the fraction by which your money increased over the year. The increase in your money is the accumulated amount minus the principal.

The fractional increase is

$$\frac{S - P}{P}$$
$$= \frac{S}{P} - 1.$$

The effective interest rate expressed as a decimal is

$$
\begin{aligned}
j_{eff} &= \frac{S}{P} - 1 \\
&= \frac{P\left(1 + \frac{j_m}{m}\right)^m}{P} - 1 \qquad \text{since} \quad S = P\left(1 + \frac{j_m}{m}\right)^m \\
&= \left(1 + \frac{j_m}{m}\right)^m - 1.
\end{aligned}
$$

From this the nominal rate j_m in terms of the effective rate j_{eff} can be determined:

$$
\begin{aligned}
j_{eff} + 1 &= \left(1 + \frac{j_m}{m}\right)^m \\
1 + \frac{j_m}{m} &= \left(j_{eff} + 1\right)^{\frac{1}{m}} \\
\frac{j_m}{m} &= \left(j_{eff} + 1\right)^{\frac{1}{m}} - 1 \\
j_m &= m\left(\left(j_{eff} + 1\right)^{\frac{1}{m}} - 1\right)
\end{aligned}
$$

Suppose we are given a nominal rate of j_m and want to find the equivalent rate j_n. Both rates must produce the same fractional increase in money over a one-year term. They must therefore both correspond to the same effective rate.

$$
\begin{aligned}
\left(1 + \frac{j_m}{m}\right)^m - 1 &= \left(1 + \frac{j_n}{n}\right)^n - 1 \\
\left(1 + \frac{j_m}{m}\right)^m &= \left(1 + \frac{j_n}{n}\right)^n \\
1 + \frac{j_n}{n} &= \left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} \\
\frac{j_n}{n} &= \left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1 \\
j_n &= n\left(\left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1\right)
\end{aligned}
$$

The nominal rate j_n compounded n times per year is equivalent to the nominal rate j_m compounded m times per year.

⌐**Example 3.5**

Find the nominal rate compounded semi-annually that is equivalent to a nominal rate of 12% per year compounded quarterly.

$$
\begin{aligned}
j_n &= n\left(\left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1\right) \\
&= 2\left(\left(1 + \frac{0.12}{4}\right)^{\frac{4}{2}} - 1\right) \\
&= 0{,}1218
\end{aligned}
$$

A nominal rate of 12% compounded quarterly is equal to 12,18% compounded semi-annually.   ⌐

3.3 Odd period calculations

After working through this section you should be able to

▷ explain the concept of an odd period and be able to express the odd period as a fraction of a whole period;

▷ calculate interest and related accumulated sums for odd periods by

    − using simple interest for the odd periods, and

    − using fractional compounding.

Up to now we have assumed that the term for any money deposited or borrowed and subject to compound interest, coincides with a whole number of compounding periods. Stated mathematically, we assumed that n is an integer.

Of course, in practice this is not necessarily the case, unless all interest is credited on a daily basis. It could well be that you wish to invest in a particular investment that credits interest on, say, the first day of each month, but that you have the money

available to do so in the middle of the month. What happens to the odd half month? Do you get no interest? Surely not!

Exactly what happens depends on the practice adopted by the particular financial institution, and you are therefore advised to find out before investing your hard-earned money. In most cases, the odd periods are treated as simple interest cases, as illustrated in the following example:

Example 3.6

An amount of R2 500 is invested on 15 May for five months in a special savings account at an interest rate of 16% per annum, compounded monthly on the first day of each month, while simple interest is applicable for the odd period calculations. How much interest is received at the end of the term?

Note that there are two odd periods involved – at the beginning and at the end. The calculation thus requires three steps:

(a) Simple interest from 15 May to 1 June

(b) Compound interest from 1 June to 1 October

(c) Simple interest from 1 October to 15 October

NB Recall the rule for counting days explained in section 2.4 – "Count the first, but not the last".

In terms of a time line, the calculation may be represented as follows:
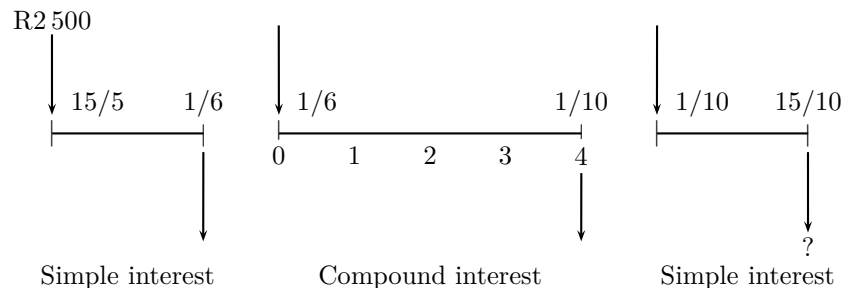
The calculation proceeds as follows:

(a) 15 May to 1 June: Day number 152 (1 June) minus day number 135 (15 May) equals 17 days, thus 17 days' simple interest.

$$
\begin{aligned}
S &= P(1 + rt) \\
&= 2\,500 \left(1 + 0{,}16 \times \frac{17}{365} \right) \\
&= 2\,518{,}63
\end{aligned}
$$

The R2 500 accumulated to R2 518,63.

(b) 1 June to 1 October: The amount of R2 518,63 is now invested for four months

at a nominal rate of 16% per annum, compounded monthly.

$$
\begin{aligned}
S &= P\left(1+i\right)^{n} \\
&= P\left(1+\frac{j_m}{m}\right)^{tm} & \left(t=\frac{4}{12},\, m=12\right) \\
&= 2\,518{,}63 \times \left(1+\frac{0{,}16}{12}\right)^{\left(\frac{4}{12}\times\frac{12}{1}\right)} \\
&= 2\,655{,}67
\end{aligned}
$$

The R2 518,63 accumulated to R2 655,67.

(c) 1 October to 15 October: Lastly, this amount earns simple interest for 14 days

$$
\begin{aligned}
S &= P(1+rt) \\
&= 2\,655{,}67\left(1+0{,}16\times\frac{14}{365}\right) \\
&= 2\,671{,}97
\end{aligned}
$$

The R2 655,67 accumulated to R2 671,97.

Finally, the interest earned is R171,97 $(2\,671{,}97-2\,500)$. It is interesting to compare the sum accumulated in this way over the five months with the sum that would have been obtained it there had been no odd periods. In other words, what sum would have been accumulated if the R2 500 had been compounded for five straight months?

$$
\begin{aligned}
S &= P\left(1+i\right)^{n} \\
&= P\left(1+\frac{j_m}{m}\right)^{tm} & \left(t=\frac{5}{12},\, m=12\right) \\
&= 2\,500\left(1+\frac{0{,}16}{12}\right)^{\left(\frac{5}{12}\times\frac{12}{1}\right)} \\
&= 2\,671{,}17
\end{aligned}
$$

The R2 500 accumulated to R2 671,17.

In other words, in this case you would receive 80c less. Not worth hassling about you may say. Maybe so. But what if you were investing R25 million on behalf of your company? How would your company auditors feel about an unnecessary loss of R8 000? They might ask you some awkward questions.

Hang on! I hear the niggling question at the back of your mind. If this order of difference can occur from such apparently trivial variations in calculation, how can we merely assume that all months are equally long? Of the four relevant months two have 30 days and two have 31 days. Will this not also be of consequence? Yes, it might. All we have to do is to calculate the cumulative effect, month by month. This involves six steps, but, since the sum accumulated at the end of each month is the principal at the beginning of the next month, we can write out the final result in a single long expression as follows:

$$
\begin{aligned}
S = 2\,500 \quad &\times \quad \left(1+0{,}16\times\tfrac{17}{365}\right) \quad \text{May} \\
&\times \quad \left(1+0{,}16\times\tfrac{30}{365}\right) \quad \text{June} \\
&\times \quad \left(1+0{,}16\times\tfrac{31}{365}\right) \quad \text{July} \\
&\times \quad \left(1+0{,}16\times\tfrac{31}{365}\right) \quad \text{August} \\
&\times \quad \left(1+0{,}16\times\tfrac{30}{365}\right) \quad \text{September} \\
&\times \quad \left(1+0{,}16\times\tfrac{14}{365}\right) \quad \text{October} \\
= 2\,672{,}35 \quad &
\end{aligned}
$$

The R2 500 accumulated to R2 672,35. This is 38c better than the result obtained in our first calculation, which confirms your suspicion that counting the days in the month matters. To be fair, nowadays, with the aid of computers, most financial institutions do indeed count the days and credit interest in this way. But, to be on the safe side, whenever you enter into a financial contract involving interest, make sure you understand the basis on which interest is calculated and credited. ⌟

After this long example, an exercise:

Exercise 3.7

Calculate the sum accumulated on a fixed deposit if R10 000 is invested on 15 March 2010 until 1 July 2012 and if interest is credited annually on 1 July at 15,5% per annum, with simple interest calculations for the odd periods.

Again, you saw above how we dealt with the odd period. In this case, first the accumulation of simple interest for 108 days and, thereafter, two years of compound interest.

I do not wish to confound the issue (as opposed to compound!) and confuse you but, with the advent of sophisticated calculators such as the recommended ones, there is yet another way of doing the calculation. This is the so-called *"fractional compounding"*, which you are likely to come across in modern textbooks on financial theory.

The point is that, instead of splitting up the calculation into two parts, why not simply take the term as a number of whole periods *plus* the relevant fractional part of the period. The fraction should be expressed in the same units as the compounding periods. Thus, if the compounding periods are quarters, the fraction should be expressed as a fraction of a quarter. *Therefore, in the previous exercise, we would have n = 2 full years plus the 108 days expressed as a fraction of a year.*

Thus $n = (2 + \frac{108}{365}) = 2{,}29589...$ years.

This may look horrendous but it is no problem for your calculator, which, you may recall, can handle fractional powers with ease.

Thus, in the previous exercise $t = $ the number of full periods as a fraction of a year plus the number of odd periods as a fraction of a year and $m = $ the number of compounding periods per year.

$$
\begin{aligned}
S &= P\left(1+i\right)^{n} \\
&= P\left(1+\frac{j_m}{m}\right)^{tm} \\
&= 10\,000(1+0{,}155)^{\left(2+\frac{108}{365}\right)} \\
&= 13\,921{,}35
\end{aligned}
$$

The sum accumulated is R13 921,35.

This is R30,72 less than the above alternative. But which is correct? Well, traditionally, the stepwise approach outlined above was the one used. However, we must admit that, with the aid of your calculator, the method of fractional compounding

is much less tedious. For this reason, such methods tend to creep into computer programs, especially since some theorists argue that these are the "exact" basis for compounding. I myself think that maybe that's changing the rules of the game, but once again, be warned: Check the fine print in your contract. (As I said, before, if interest is calculated on a daily basis, as is done by many financial institutions, then the problem essentially disappears.)

Exercise 3.8

Compare the amounts accumulated on a principal of R5 000 for five years and three months at 18% per annum if

(a) simple interest is used for the odd period and compound interest for the rest of the term

(b) fractional compounding is used for the full term

Mathematical note

If you are mathematically curious, it is interesting to investigate the difference between the two approaches. (If you are not curious skip to section 3.4 – the remainder of this section is not earmarked for examination purposes.) To do this, suppose $P = 1$ (since it is not relevant to the argument), that the interest rate per period is r and that the term can be split into an integral part and a fractional part as follows

$$n = k + f$$

with k the integer and f a fraction, such that $0 < f < 1$.

Then the length of the odd period will be f, and if simple interest is earned over this period, the principal will accrue to

$$
\begin{aligned}
S &= P(1 + rf) \\
&= 1 + rf. \qquad (P = 1)
\end{aligned}
$$

During the rest of the term, this amount will increase by the accumulated amount earning compound interest.

$$
\begin{aligned}
S_1 &= P(1 + rf)\left(1 + \frac{j_m}{m}\right)^{tm} \qquad \left(\text{but } tm = n, \ \frac{j_m}{m} = r \text{ and } P = 1\right) \\
&= (1 + rf)(1 + r)^n.
\end{aligned}
$$

If fractional compounding is used, the principal amount will increase to:

$$
\begin{aligned}
S_2 &= P\left(1 + \frac{j_m}{m}\right)^n \\
&= (1 + r)^{k+f} \qquad \left(P = 1; \ \ n = k + f; \ \ r = \frac{j_m}{m}\right) \\
&= (1 + r)^f (1 + r)^k.
\end{aligned}
$$

(The last step follows from the laws for combining exponents ($a^m a^n = a^{m+n}$).)

In other words, the difference between the two cases can be attributed to the difference between the two factors $(1 + rf)$ and $(1 + r)^f$.

Now, with the help of Taylor's Theorem, it is possible to show that $(1 + r)^f$ can be written as a series. (You will have to take my word for this. If not, you will have to take a more advanced maths course!) The series is

$$(1 + r)^f = 1 + rf + \frac{1}{2}r^2 f(f - 1) + \ldots$$

In words:

$$(1 + r)^f = (1 + rf) + \text{ correction terms.}$$

The question is thus: How big are the correction terms, since they are merely the difference between the two cases? We can examine this by looking at a few values. Obviously, because of the factor $f(f - 1)$, if f is close to 0 or 1 (ie near the beginning or end of a period), the correction will be very small for all values of r. So, lets take the worst case, namely $f = \frac{1}{2}$ (ie the middle of a period). Now suppose that interest rates are quite high, say, $r = 25\%$ per period. Then the correction is

$$\begin{aligned}\frac{1}{2}r^2 f(f - 1) &= \frac{1}{2} \times (0{,}25)^2 \times \frac{1}{2} \times (-\frac{1}{2}) \\ &\approx -0{,}008\end{aligned}$$

Compare this with $1 + rf = 1 + 0{,}25 \times \frac{1}{2} = 1{,}125$.

Therefore $(1 + r)^f \approx 1{,}125 - 0{,}008 = 1{,}117$. This explains why the interest earned in the second case is less than the first. (You may be wondering about the other correction terms. Don't! They are even smaller and together contribute about $+0{,}0005$ in this case.)

On the other hand, if interest rates are low, say 5% per period, we find the following:

$$(1 + rf) = 1 + 0{,}05 \times \frac{1}{2} = 1{,}025$$

and

$$\frac{1}{2}r^2 f(f - 1) = \frac{1}{2} \times (0{,}05)^2 \times \frac{1}{2} \times (-\frac{1}{2}) = -0{,}0003.$$

We see that the difference is now only in the fourth decimal.

In other words, we can conclude that for low interest rates, the differences between the two cases are relatively minor. This explains a lot. It is only in the last few decades that the world has seen the introduction of interest rates high enough to make the differences matter. Previously, such differences could be ignored; hence they were not discussed in older financial textbooks. Also, what is more relevant for us today is that if the compounding period is small say, daily, the interest rate per period will be small, and again, the differences will not be of much importance. But, as always, it all depends on what one regards as important. If you are a big-shot financial investor, playing with millions on the financial market, small differences could just be important to you! So it's just as well you understand the difference.

3.4   Continuous compounding

<div style="border:1px solid #000; padding:10px;">

After working through this section you should be able to

▷ explain the concept underlying the introduction of continuous compounding;

▷ apply and know the formulæ relating continuous rates and nominal rates for different compounding periods;

▷ use continuous rates to choose between investment options with different nominal rates and different compounding periods.

</div>

Study

unit

3.4

I would like you to return with me to example 3.4 on page 31 and examine the results carefully. You will recall that we calculated the effective rates of interest when the nominal rate was 15% per annum and the interest was calculated for shorter and shorter periods. Specifically, let us plot the effective interest rate as a function of the number of compounding periods per year:
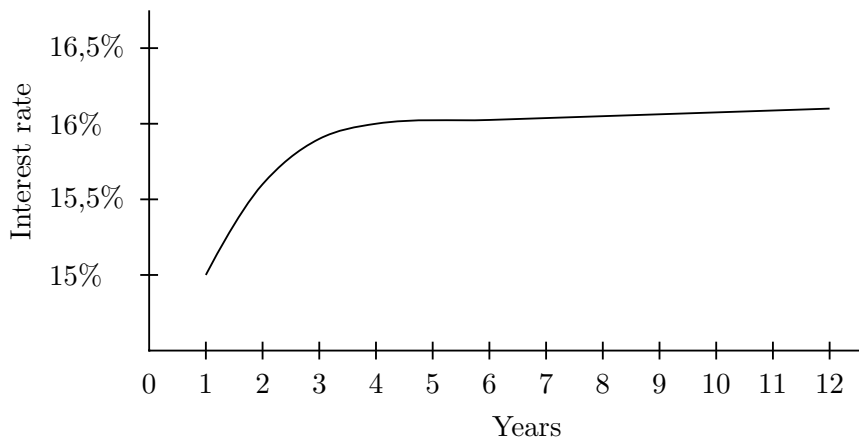
It seems that the more often interest is compounded, the larger the effective interest rate becomes. It would also seem, however, that as the number of compounding periods increases, so the effective interest rate gets closer and closer to the line at about 16,2%. In fact you will recall that, for daily compounding

$$
\begin{aligned}
S &= 100 \times \left(1 + \frac{0{,}15}{365}\right)^{365} \\
&= 116{,}1798.
\end{aligned}
$$

The accumulated amount is R116,1798. Suppose we were to compound twice each day (say, in the morning and in the evening).

$$
\begin{aligned}
S &= 100 \times \left(1 + \frac{0{,}15}{365 \times 2}\right)^{365 \times 2} \\
&= 116{,}1816
\end{aligned}
$$

The accumulated amount is R116,1816. Or hourly!

$$
\begin{aligned}
S &= 100 \times \left(1 + \frac{0{,}15}{365 \times 24}\right)^{365 \times 24} \\
&= 116{,}1833
\end{aligned}
$$

The accumulated amount is R116,1833. It does indeed seem as if we are getting close to some "limiting" value of 16,18...%. Examine the results for exercises 3.2 and 3.6 and you will find the same effect.

Can we prove that such a limiting value does indeed exist? And why is it important to know that it exits?

Let's first answer the second question. If such a limiting value did not exist, it would have meant that the future value of an investment (or debt) could be made arbitrarily large by increasing the compounding frequency. If it does exist, we know that there is an upper limit to the accrued value of an investment or debt over a limited time period.

Let us now try to answer the first question.

Look again at the expression for compounding m times per year with a nominal annual rate of j_m. Then, in one year, where $t = 1$,

$$
S = P\left(1 + \frac{j_m}{m}\right)^m .
$$

What happens when m gets larger and larger – that is, when we compound interest more and more frequently?

Now, remember in the note to exercise 3.3 we defined the number e as the value of the expression

$$
\left(1 + \frac{1}{m}\right)^m
$$

where m is a number that is so large that it approaches infinity.

Right now, we are considering the expression

$$
\left(1 + \frac{j_m}{m}\right)^m
$$

where m is a number that is so large that it approaches infinity. Could this possibly be related to the number e? It could indeed! The details of proving the relationship involves mathematics that is beyond the scope of this study guide. Therefore you may accept that it can be proved mathematically that the equation

$$
\left(1 + \frac{j_m}{m}\right)^m
$$

may be replaced by

$$
\begin{aligned}
&\lim_{m \to \infty} \left(1 + \frac{j_m}{m}\right)^m \quad \left[\lim_{n \to \infty}\left(1 + \frac{a}{n}\right)^n = e^a\right] \\
&= \; e^{j_m}
\end{aligned}
$$

41

when m is a number that is so large that it approaches infinity. We can then write

$$S = Pe^{jm}.$$

To find the value of the associated effective interest rate, we write, as in section 3.2:

$$
\begin{aligned}
I &= S - P \\
&= Pe^{jm} - P \\
&= P(e^{jm} - 1).
\end{aligned}
$$

But since $P = 100$, we can write the effective interest rate (as a percentage) as:

$$J_{eff} = 100(e^{jm} - 1).$$

This is the effective interest rate when the number of compounding periods tends to infinity. Therefore we will use the symbol J_∞ to identify it, and now define

$$J_\infty = 100(e^{jm} - 1).$$

Now use your calculator to find the value of J_∞ for $j_m = 0,15$. We get

$$
\begin{aligned}
J_\infty &= 100(e^{0,15} - 1) \\
&= 16,18342427\%.
\end{aligned}
$$

Note that the value for hourly compounding differed from this by only one digit in the fourth decimal place.

Note

In future, we will refer to the case where interest is compounded an almost infinite number of times as **continuous compounding** at a rate c, and to J_∞ as the effective interest rate expressed as a percentage for continuous compounding.

Thus

$$J_\infty = 100(e^c - 1).$$

Exercise 3.9

Determine the effective interest rate for continuous compounding where $c = 22\%$ (which was used in exercise 3.6 in section 3.2).

Thus, finally, with continuous compounding at rate c and for principal P, we can write that:

The sum accumulated in *one year* is

$$S = Pe^c.$$

NB

The sum accumulated in *t years* will then simply be

$$S = P(e^c)^t = Pe^{ct}.$$

"Yes, but what is the practical value of all this?", I hear you mutter. Well, the use of continuous compounding is mathematically much more simple. For example, in the above equation, the number of years' t does not have to be a whole number, but may be any decimal number. Or, suppose an amount P is invested at a nominal annual rate of j_1 for term t_1 and that the sum accumulated is then immediately reinvested at rate j_2 for a term t_2. The sum accumulated over the full period $t_1 + t_2$ is thus

$$S = Pe^{j_1 t_1} e^{j_2 t_2}.$$

But, using the laws of combining exponents, this can be written as

$$S = Pe^{j_1 t_1 + j_2 t_2}.$$

Simplifications of this type make continuous compounding particularly useful in financial theory. Also, as we shall see in a moment, it gives us a particularly useful basis for comparing different investment opportunities with different compounding periods and rates.

To do this, we must relate ordinary compounding to continuous compounding. In section 3.2 we obtained an equation to find the effective interest rate j_{eff} for any given compounding period and the nominal interest rate j_m. We wish to derive a similar relationship for the continuous compounding rate, which we shall call c. (In many books you will find c referred to as r, but this would confuse the issue here where we have reserved r for the rate per period.)

Suppose that we have a sum P that we invest for one year, on the one hand, at a nominal annual rate of j_m compounded m times per year and, on the other, at a continuous compounding rate of c. In these two cases the sum accumulated in one year is then respectively

$$S = P \left(1 + \frac{j_m}{m}\right)^m$$

and

$$S = Pe^c.$$

The question is: What must the continuous rate c be for these two amounts to be equal? It must be

$$e^c = \left(1 + \frac{j_m}{m}\right)^m$$

since the principal is the same in both cases.

We can solve for c by taking the natural logarithm of both sides. (Refer to the definition of the natural logarithm in section 3.1.)

Now

$$\ln e^c = \ln\left(\left(1 + \frac{j_m}{m}\right)^m\right)$$
$$c \ln e = m \ln\left(1 + \frac{j_m}{m}\right). \qquad (\ln e = 1)$$

Therefore

$$c = m \ln\left(1 + \frac{j_m}{m}\right).$$

Alternatively we could express j_m in terms of c as follows.

We have

$$\left(1 + \frac{j_m}{m}\right)^m = e^c.$$

Take the mth root of both sides:

$$1 + \frac{j_m}{m} = e^{\frac{c}{m}}.$$

Thus

$$\frac{j_m}{m} = e^{\frac{c}{m}} - 1$$

$$j_m = m \times \left(e^{\frac{c}{m}} - 1\right).$$

Another way to derive these same two formulæ would be to equate the effective interest rate, J_{eff}, when the nominal annual interest rate j_m is compounded m times per year, to the effective interest rate (expressed as a percentage) for continuous compounding at the nominal annual interest rate c.

Thus with

$$J_{eff} = 100 \left(\left(1 + \frac{j_m}{m}\right)^m - 1\right)$$

and

$$J_\infty = 100(e^c - 1).$$

and if $J_{eff} = J_\infty$, then it follows that

$$\left(1 + \frac{j_m}{m}\right)^m = e^c$$

as shown.

We can now use the two equations

$$c = m \ln\left(1 + \frac{j_m}{m}\right)$$

and

$$j_m = m \left(e^{\frac{c}{m}} - 1\right)$$

to convert a continuous compounded interest rate to an equivalent nominal interest rate that is compounded periodically, or vice versa. The two rates obtained in this way are equivalent in the sense that they will yield the same amount of interest, or give rise to the same effective interest rate.

Example 3.7

You are quoted a rate of 20% per annum compounded semi-annually. What is the equivalent continuous compounding rate?

Using the first formula

$$\begin{aligned}
c &= m \ln\left(1 + \frac{j_m}{m}\right) \\
&= 2 \ln\left(1 + \frac{0,20}{2}\right) \\
&= 0,1906204 \\
&= 19,06204\%.
\end{aligned}$$

The equivalent continuous compounding rate is 19,06204% per annum.

Exercise 3.10

Suppose R12 000 was invested on 15 November at a continuous rate of 16%. What would the accumulated sum be on 18 May of the following year? (Count the days exactly.)

Earlier I pointed out that continuous rates could be used to compare investment alternatives. The next exercise illustrates this point.

Exercise 3.11

You have two investment options:

(a) 19,75% per annum compounded semi-annually

(b) 19% per annum compounded monthly

Use continuous rates to decide which is the better option.

Study
unit
3.5

3.5   Equations of value

> *After working through this section you should be able to*
>
> ▷ give and apply the equations and the corresponding time lines relating present and future values of money when compounding is applicable;
>
> ▷ state and apply the two rules for moving money backward and forward in time;
>
> ▷ use the rules to replace one set of financial obligations with another, that is, reschedule debts.

From time to time, a debtor (the guy who owes money) may wish to replace his set of financial obligations with another set. On such occasions, he must negotiate with his creditor (the guy who is owed money) and agree upon a new due date, as well as on a new interest rate. This is generally achieved by evaluating each obligation in terms of the new due date, and equating the sum of the old and the new obligations on the new date. The resultant *equation of value* is then solved to obtain the new future value that must be paid on the new due date.

It is evident from these remarks that the time value of money concepts must play an important role in any such considerations, even more so than they did in the simple interest case, since the investment terms are generally longer in cases where compound interest is relevant. For this reason, we shall first review the present and future value concepts.

You will recall that if P is invested at an interest rate of i per period for a term of length n periods, it accumulates to

$$S = P(1 + i)^n$$

and that we call P the *present value* and S the *future value* of the investment.

Sometimes we are given the future value S and wish to know the present value P. To do this, we can rearrange the above formula as follows:

$$\begin{aligned} P &= \frac{S}{(1+i)^n} \\ &= S(1+i)^{-n} \end{aligned}$$

or

$$P = S\left(1 + \frac{j_m}{m}\right)^{-(tm)}$$

If we replace the symbols P with PV and S with FV to emphasise the time value aspect, these two formulæ become

$$FV = PV(1+i)^n$$

and

$$PV = FV(1+i)^{-n}.$$

Note

In the latter case of finding the present value of some future value, PV is often referred to as the *discounted value* of the future sum, and the factor $(1+i)^{-n}$ is termed the *discount factor*. This should not be confused with the simple discounting used in the previous chapter, and for which a discount rate was defined. The rate used in compound discounting is simply the relevant interest rate. It would be possible to introduce a compound discount rate, but this is not done in practice since it does not have a readily understood meaning. The important concepts are those of present and future value, and the fact that the process of determining present value is often referred to as discounting.

Exercise 3.12

You decide now that when you graduate in four years time you are going to treat yourself to a car to the value of R120 000. If you can earn 15% interest compounded monthly on an investment, calculate the amount that you need to invest now.

As was the case for simple interest, we can summarise the above formulæ succinctly by means of a time line.

$P(1+i)^{-n_1}$ ... P

n_1 at i n_2 at i

earlier date   current date (n0w)   later date

P

$P(1+i)^{n_2}$
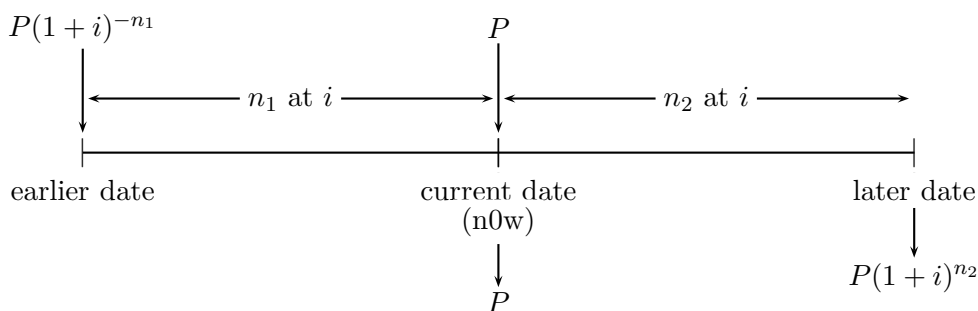
Again, we can state the results as two simple rules:

NB

- To move money forward, multiply the amount by the factor $(1+i)^n$

- To move money backward, multiply the amount by the factor $(1+i)^{-n}$

⌐ **Example 3.8**

An obligation of R50 000 falls due in three years' time. What amount will be needed to cover the debt if it is paid

(a) in six months from now

(b) in four years from now

if the interest is credited quarterly at a nominal rate of 12% per annum?

Draw the relevant time line.
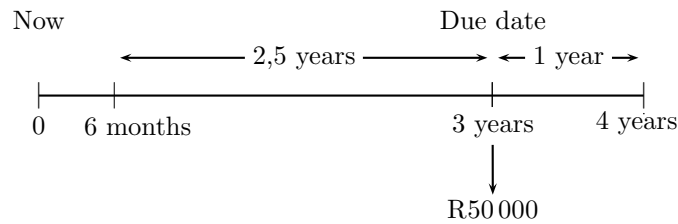
(a) To determine the debt if it is paid in six months (ie two quarters), we must discount the debt back two-and-a-half-years from the due date to obtain the amount due.

$$\begin{aligned}
S &= P(1+i)^n \\
P &= S(1+i)^{-n} \\
P &= S\left(1+\frac{j_m}{m}\right)^{-tm}
\end{aligned}$$

with $m = 4$, $t = 2,5$ and $j_m = 0,12$.

$$\begin{aligned}
P &= 50\,000 \times \left(1+\frac{0,12}{4}\right)^{-(2,5\times 4)} \\
&= 37\,204,70
\end{aligned}$$

The present value of the debt six months from now is R37 204,70.

(b) To determine the debt, if it is allowed to accumulate for one year past the due date, we must move the money forward one year to obtain.

$$\begin{aligned}
S &= 50\,000 \times \left(1+\frac{0,12}{4}\right)^{(1\times 4)} \\
&= 56\,275,44
\end{aligned}$$

The future value of the debt four years from now is R56 275,44.

⌐

Note

It is interesting to note the continuous form of the above results. In such cases, the two equations for future and present values are respectively

$$
\begin{aligned}
FV &= PVe^{ct} \\
PV &= FVe^{-ct}.
\end{aligned}
$$

In the rest of this section we shall confine ourselves to periodic (as opposed to continuous) compounding.

I said above that we would concern ourselves here with replacing one set of financial obligations with another equivalent set. This sounds complicated, but is really just a case of applying the above rules for moving money back and forward, keeping a clear head and remembering *that, at all times*, the *only money* that may be *added together* (or subtracted) is that with *a common date*. An example will illustrate the process.

Example 3.9

Trevor Sithole foresees cash flow problems ahead. He borrowed R10 000 one year ago at 15% per annum, compounded semi-annually and due six months from now. He also owes R5 000, borrowed six months ago at 18% per annum, compounded quarterly and due nine months from now. He wishes to pay R4 000 now and reschedule his remaining debt so as to settle his obligations 18 months from today. His creditor agrees to this, provided that the old obligations are subject to 19% per annum compounded monthly for the extended period. It is also agreed that the R4 000 paid now will be subject to this same rate of 19% per annum compounded monthly for evaluation purposes. What will his payment be in 18 months' time?

First we calculate the maturity values of his present obligations.

They are:

R10 000, with $j_m = 0{,}15$, $m = 2$ and $t = 1{,}5$. The maturity date is six months from now because the R10 000 was borrowed a year ago for 18 months.

$$
\begin{aligned}
S &= P(1+i)^n \\
&= P\left(1+\frac{j_m}{m}\right)^{tm} \\
S &= 10\,000\left(1+\frac{0{,}15}{2}\right)^{(1{,}5\times 2)} \\
&= 12\,422{,}97
\end{aligned}
$$

The future value of the R10 000 is R12 422,97 from now.

On R5 000, with $j_m = 0{,}18$, $m = 4$ and $t = \frac{15}{12}$ (nine plus six months):

$$
\begin{aligned}
S &= 5\,000\left(1+\frac{0{,}18}{4}\right)^{\left(\frac{15}{12}\times\frac{4}{1}\right)} \\
&= 6\,230{,}91
\end{aligned}
$$

The future value of the R5 000 nine months from now is R6 230,91.

Suppose the payment that must be made to settle all debts at month 18 from now is X. We use the time line, as shown below, and indicate all old obligations above the line and the two newly scheduled payments below.

Thus, above the line we have, at month six, the amount due then, namely R12 422,97, and, at month nine, R6 230,91. Below the line, we have the R4 000 paid now (at month zero) and the settlement X at month 18.
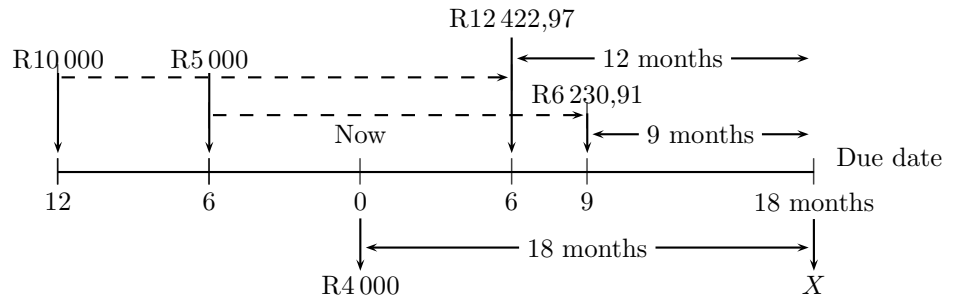
We must now evaluate each item, obligation or payment on the due date in order to determine the future value on that date. That is, we move all moneys *forward* to the due date using the agreed-upon rate of 19% per annum, compounded monthly.

Obligations:

$$\text{R12 422,97}: \quad S \;=\; 12\,422{,}97 \times \left(1 + \tfrac{0{,}19}{12}\right)^{(12\times 1)} \qquad \left(j_m = 0{,}19,\; m = 12,\; t = 1\right)$$
$$\;=\; 15\,000{,}13$$

The future value is R15 000,13.

$$\text{R6 230,91}: \quad S \;=\; 6\,230{,}91 \times \left(1 + \tfrac{0{,}19}{12}\right)^{\left(\frac{9}{12}\times\frac{12}{1}\right)} \qquad \left(j_m = 0{,}19,\; m = 12,\; t = \tfrac{9}{12}\right)$$
$$\;=\; 7\,177{,}18$$

The future value is R7 177,18.

Payments:

$$\text{R4 000}: \quad S \;=\; 4\,000\left(1 + \tfrac{0{,}19}{12}\right)^{\left(\frac{18}{12}\times\frac{12}{1}\right)} \qquad \left(j_m = 0{,}19,\; m = 12,\; t = \tfrac{18}{12}\right)$$
$$\;=\; 5\,307{,}19$$

The future value is R5 307,19.

X is the amount to be paid on the due date and is therefore not subject to interest.

We have all moneys expressed in terms of values on the due date. Equate the sum of the obligations to the payments on the due date, that is

$$\text{Total obligations} = \text{Total payments (as on the due date)}.$$

This is the equation of value that gives

$$15\,000{,}13 + 7\,177{,}18 = 5\,307{,}19 + X$$

or

$$X \;=\; 15\,000{,}13 + 7\,177{,}18 - 5\,307{,}19$$
$$\;=\; 16\,870{,}12.$$

Tenesmus must pay R16 870,12 in 18 months' time, in order to settle his debts (and buy time). I hope he knows what he is letting himself in for!

Note that the obligations can only be subject to the new interest rate after their original due dates. Until such dates, they are subject to the old (contracted) interest rates.

As I said, the principles are quite simple and you should not be overwhelmed by their apparent complexity. Let us do another example just to make sure that you understand them.

Example 3.10

Peter Penniless owes Wendy Worth R5 000 due in two years from now, and R3 000 due in five years from now. He agrees to pay R4 000 immediately and settle his outstanding debt completely three years from now. How much must he pay then if they agree that the money is worth 12% per annum compounded half-yearly?

Since the compound amounts owed on the respective due dates are given, we don't need to calculate these. Consequently, we can draw the relevant time line immediately.
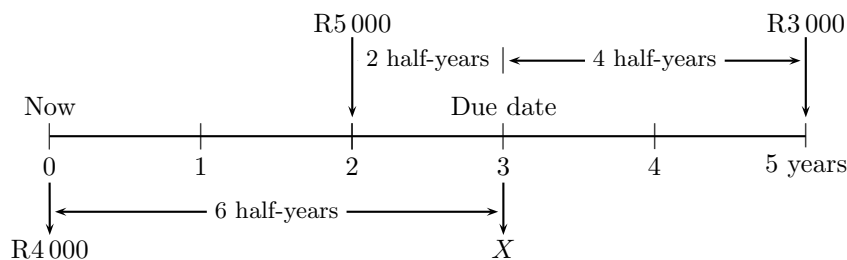
We determine the values of each item on the new due date of three years hence.

Obligations:

$$\text{R5 000}: \quad S \ = \ 5\,000 \times \left(1 + \frac{0{,}12}{2}\right)^{(1 \times 2)}$$
$$= \ 5\,000 \times 1{,}06^2$$
$$\text{R3 000}: \quad P \ = \ 3\,000 \times \left(1 + \frac{0{,}12}{2}\right)^{(-2 \times 2)}$$
$$= \ 3\,000 \times 1{,}06^{-4}.$$

Note that, since the obligation of R3 000 is actually paid earlier than its due date, we have to perform a "present-value" type of calculation – that is, bring it back.

Payments:

$$\text{R4 000}: \quad S \ = \ 4\,000 \times \left(1 + \frac{0{,}12}{2}\right)^{(3 \times 2)}$$
$$= \ 4\,000 \times 1{,}06^6$$
$$X: \quad \text{No interest.}$$

Equating obligations and payments gives

$$X + 4\,000 \times 1{,}06^6 = 5\,000 \times 1{,}06^2 + 3\,000 \times 1{,}06^{-4}$$

or

$$
\begin{aligned}
X &= 5\,000 \times 1{,}06^2 + 3\,000 \times 1{,}06^{-4} - 4\,000 \times 1{,}06^6 \\
&= 5\,618 + 2\,376{,}28 - 5\,674{,}08 \\
&= 2\,320{,}20
\end{aligned}
$$

Thus, in three years' time, Peter must pay Wendy R2 320,20.   ⌐

Why do we choose the due date for calculation purposes? The reason is that it is the obvious one. But does it matter? No, any comparison date could be chosen. Suppose we had taken the due date for the R3 000 (in five years) as the comparison date, as indicated on the following time line:
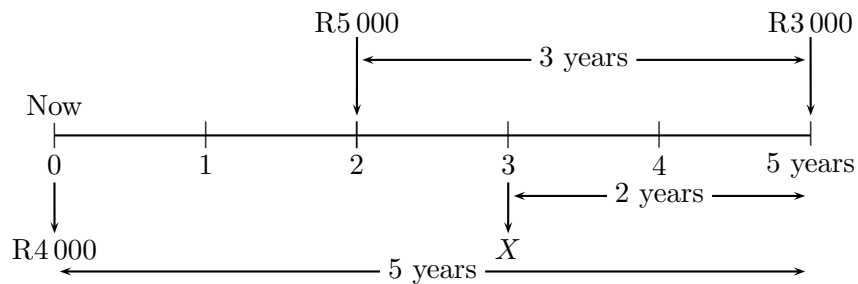
In this case, the appropriate equation of value is (check!)

$$
X \times 1{,}06^4 + 4\,000 \times 1{,}06^{10} = 5\,000 \times 1{,}06^6 + 3\,000.
$$

But this is simply the first equation of value above multiplied by $1{,}06^4$. In other words, the solution, that is, the value of X, will be the same. In fact, by multiplying $1{,}06^n$ for different values of n we can set up the equation of value for any comparison date we like. In all cases, the solution is the same. In practice, it is best to select the due date as that of the final payment, since it is the most direct. The particular date that we choose for formulating our *equation of value* is called the *focal date*, *valuation date* or *comparison date*. Of course, if the dates of the payments are shifted, say for example, X is to be paid after four years, then the solution (ie the value of X) will change.

Try the following exercise now. It is slightly more complicated since you must first calculate the maturity values of the various debts and, in addition, more than one payment must be made. But just keep your cool and draw your time line, and you should manage.

Exercise 3.13

Three years ago Fiona Fin borrowed R4 000 from Martin Moneylender for five years at 12% per annum compounded monthly. One year ago she borrowed R8 000 at 16% per annum compounded quarterly, for five years. She agrees to repay her debt in two equal instalments, one now and one in five years' time from now. If Martin's money is now worth 20% per annum compounded half-yearly, what is the amount of each payment?

Finally, we summarise the *equation of value procedure*:

1. Draw a clear time line showing the dated values of all debts above the line and the dated values of payments below the line. (These must include any unknowns – usually payments – as symbols, eg X.)

2. Select the comparison date and bring all moneys back or forward to this date using the specified interest rate.

3. Set up the equation of value. The sum of all the payments equals the sum of all the obligations on the comparison date.

4. Solve the equation algebraically.

3.6   Summary of Chapter 3

In this chapter, the basic concepts and formulæ pertaining to *compound interest* were reviewed. The basic formula is

$$S = P(1 + i)^n \ \ \text{or} \ \ S = P\left(1 + \frac{j_m}{m}\right)^{tm}$$

where
$$
\begin{aligned}
S &\equiv \text{the accrued amount}\\
P &\equiv \text{the initial principal}\\
i &\equiv \tfrac{j_m}{m} = \text{the annual interest rate compounded } m \text{ times per year}\\
n &\equiv t \times m\\
j_m &\equiv \text{the annual interest rate}\\
t &\equiv \text{the number of years' of investment}\\
m &\equiv \text{the number of compounding periods per year}
\end{aligned}
$$

Conversely, the *present value* is given by

$$
\begin{aligned}
P &= \frac{S}{(1+i)^n} & : \qquad P &= \frac{S}{\left(1+\frac{j_m}{m}\right)^{tm}}\\
&= S(1+i)^{-n} & &= S\left(1+\frac{j_m}{m}\right)^{-tm}
\end{aligned}
$$

It was also shown how i, j_m, n or t could be determined in the case of compound interest when S and P are given.

The concept of effective interest rate for nominal annual compound interest rates when the period of compounding is less than one year, was discussed. If the *nominal annual* rate is j_m, and compounding occurs m times per year, then the *effective rate* is given by J_{eff}, where

$$J_{eff} = 100\left(\left(1 + \frac{j_m}{m}\right)^m - 1\right) \ \text{(expressed as a percentage)}$$

or

$$j_{eff} = \left[\left(1 + \frac{j_m}{m}\right)^m - 1\right] \quad \text{(expressed as a decimal)}$$

Odd period calculations and the concept of *fractional compounding* were discussed, and it was shown how to deal with these. *Continuous compounding* was explained, and it was shown that, in this case, the compounding formula becomes

$$S = Pe^{ct} \text{ and, inversely, } P = Se^{-ct}$$

where c is the continuous compounding rate. Formulæ for relating the continuous rate c to the nominal rate j_m for compounding with m periods per year, were derived:

$$\begin{aligned} c &= m\ln\left(1 + \frac{j_m}{m}\right) \\ j_m &= m(e^{\frac{c}{m}} - 1). \end{aligned}$$

The use of continuous rates for comparing alternative instruments was illustrated.

The conversion of one nominal rate to another nominal rate was discussed and the formula derived with

$$j_n = n\left(\left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1\right).$$

Equations of value, using compound interest, were introduced as a basis for rescheduling debt, and were illustrated by way of examples. Allied to this was the concept of a time line. This was shown to be a powerful conceptual tool for handling interest-related problems.

3.7   Evaluation exercises

1. Calculate the accrued principal on R3 000 invested at 12% per annum for three years and compounded monthly.

2. How much must be invested at 16% per annum, if interest is compounded quarterly, to yield R10 000 in two-and-a-half years?

3. What is the rate of interest if R4 000 yields R6 000 in three years and interest is compounded monthly?

4. Determine the effective rates of interest (express as a percentage) if the nominal rate is 18% and interest is calculated

   (a) half-yearly   (b) monthly.

5. Compare the amounts accumulated on a principal of R10 000 is invested from 10 March 2010 to 1 July 2012 at $16\frac{1}{2}$% per annum compounded semi-annually, and credited on 1 January and 1 July, if

   (a) simple interest is used for the odd period and compound interest for the rest of the term;

53

(b) fractional compounding is used for the full term.

Note: For this question you may assume that each half year is indeed 0,5 years (ie ignore the slight differences between the number of days).

6. (a) What nominal rate for quarterly compounding is equivalent to 12% per annum compounded monthly?

   (b) What nominal rate for daily compounding is equivalent to 18% per annum compounded semi-annually?

7. Paul Poverty owes Winston Wealth R1 000 due in three years from now and R8 000 due in five years from now. He wishes to reschedule his debt so as to pay two sums on different dates, one, say X, in one year from now, and the other, which is twice as much (ie $2X$), five years later. Winston agrees, provided that the interest rate is 18% per annum, compounded quarterly. What are Paul's payments?

Annuities

Outcome of chapter

To master the basic concepts and applications of annuities.

Key concepts

- ✓ Annuity
- ✓ Payment interval or period
- ✓ Term
- ✓ Ordinary annuity
- ✓ Annuity due
- ✓ Annuity certain

- ✓ Perpetuity
- ✓ Amount or future value
- ✓ Present value
- ✓ Deferred annuity
- ✓ General annuity
- ✓ Increasing annuity

4.1   Basic concepts

After working through this section you should be able to

▷ explain the basic structure and elements of an annuity;

▷ define and give examples of an ordinary annuity certain, an annuity due and a perpetuity.

Definition 4.1 *An annuity is a sequence of equal payments at equal intervals of time.*

Examples of annuities include the annual premium payable on a life insurance policy, interest payments on a bond and quarterly stock dividends.

The *payment interval* (or *period*) of an annuity is the time between successive payments, while the *term* is the time from the beginning of the first payment interval to the end of the last payment interval. These concepts are illustrated by the following time line (for payment R):
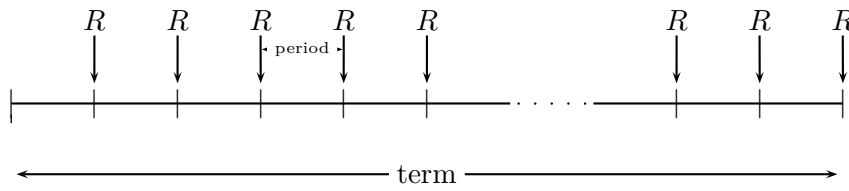
An annuity is termed an *ordinary* annuity when payments are made at the same time that interest is credited, that is, at the *end* of the payment intervals. By contrast, an annuity with a periodic payment that is made at the *beginning* of each payment interval is known as an *annuity due.*

If the payments begin and end on a fixed date, the annuity is known as an *annuity certain.* On the other hand, if the payments continue for ever, the annuity is known as *perpetuity.*

The above three cases are represented by the following time lines:

Ordinary annuity certain
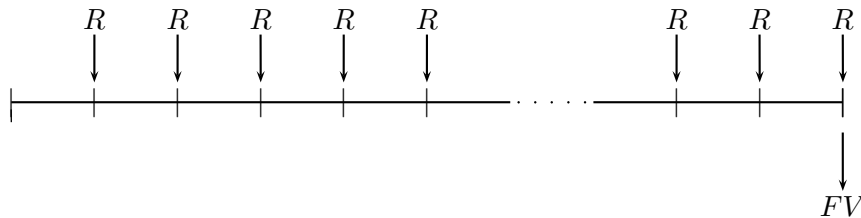
Annuity certain due
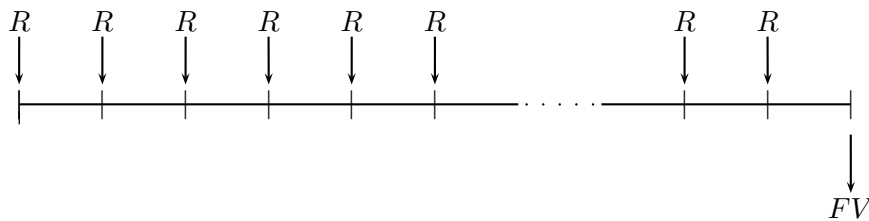
Perpetuity
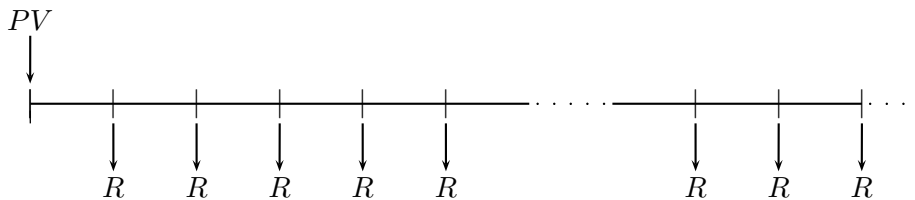
The *amount* or *future value* of an annuity is the sum of all payments made and the accumulated interest at the end of the term.

The *present value* is the sum of payments, each discounted to the beginning of the term, that is, the sum of the present value of all payments.

The basic concepts are illustrated by the following examples (notice how time lines, with the interest period as the unit, are used).

Example 4.1

The premium on an endowment policy is R240 per year payable for 20 years, and payable at the end of each period.

This is represented on a time line.
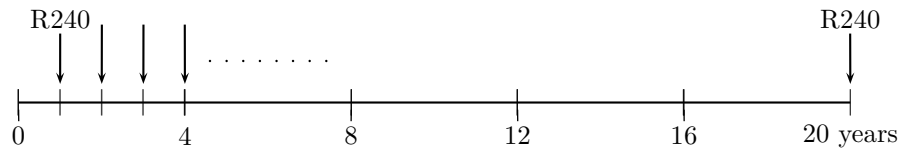
Here the payment interval is one year and the term is 20 years. Since the payments are made at the end of each period, this is an ordinary annuity certain.

Example 4.2

The monthly rent for a shop is R1 000 payable in advance.

This is represented on a time line.

Here, the payment interval is one month, while the term will be determined by the contract. Seeing that nothing else is stated, we assume that it is one year. Since the payment is made at the beginning of each period, this is an annuity due.

Example 4.3

A company is expected to pay R1,80 indefinitely every six months on a share of its preferred stock.

This is an example of a perpetuity with a payment interval of six months. In this case, no term is defined since payment is indefinitely.

4.2   Ordinary annuities certain

After working through this section you should be able to

▷ derive and explain each step of the derivation of the formulæ for the present and future values of an ordinary annuity certain;

▷ apply the formulæ for the present and future values of an ordinary annuity certain.

In this section we shall concentrate mainly on the ordinary annuity certain, since the basic mathematics and calculations are best described in terms of it. In the next section we look at the annuity due in more detail. From now on, we shall simply speak of "**an annuity**", which, **unless otherwise stated, will mean an ordinary annuity certain**.

We first look at an example in which the accumulated value of an annuity is calculated, and then derive a general formula for the accumulated value.

⌈**Example 4.4**

Determine the accumulated value of an annuity after four payments, each R1 000 paid annually, at an interest rate of 12% per annum.

At the end of the term, the first payment of R1 000 will have accumulated interest for three years compounded at 12% per annum as indicated by the following time line:
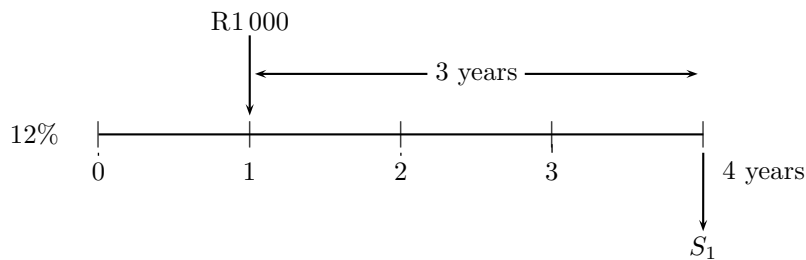
$$
\begin{aligned}
S &= P(1+i)^n \\
S_1 &= 1\,000(1+0{,}12)^3 \\
&= 1\,404{,}93
\end{aligned}
$$

Its accumulated value, S_1, at the end of the term is R1 404,93. This is obviously a straightforward application of the compound interest formula.

Similarly, the accumulated values of the second, third and fourth payments at the end of the term are respectively:

$$
\begin{aligned}
S_2 &= 1\,000(1+0{,}12)^2 = 1\,254{,}40 \\
S_3 &= 1\,000(1+0{,}12)^1 = 1\,120{,}00 \\
S_4 &= 1\,000
\end{aligned}
$$

Thus

$$
\begin{aligned}
S &= S_1 + S_2 + S_3 + S_4 \\
&= 1\,000(1 + 0{,}12)^3 + 1\,000(1 + 0{,}12)^2 + 1\,000(1 + 0{,}12) + 1\,000 \\
&= 4\,779{,}33.
\end{aligned}
$$

The amount of the annuity after four payments is R4 779,33. The result is represented by the following time line:
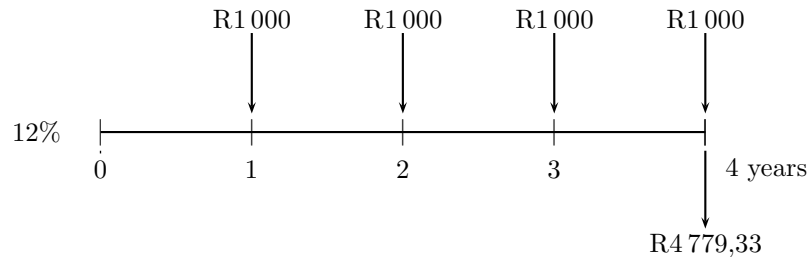
Exercise 4.1

Determine the amount (future value) of an annuity after five payments of R600, each paid annually, at an interest rate of 10% per annum.

The pattern in this type of calculation should already be evident. Let us see if we can derive a formula for calculating the amount of an annuity in general.

It is obvious from both the above exercise and the example that the size of the payment can be factored out in each case (using the distributive law). We therefore assume that each payment is R1.

Furthermore, suppose that the interest rate is i per period and that the number of periods in the term is n. The first payment of R1 will be compounded over $n - 1$ periods, as shown by the following time line
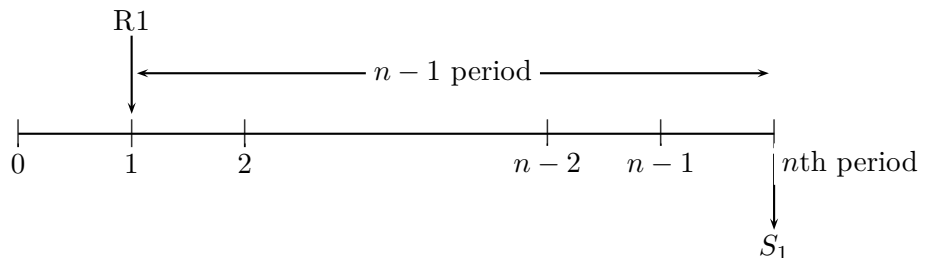
and will accumulate to the value of S_1 at the end of the term where

$$
S_1 = 1(1 + i)^{n-1}.
$$

The second payment will accumulate to S_2, where

$$
S_2 = 1(1 + i)^{n-2}.
$$

And so on

$$S_3 \;=\; 1(1+i)^{n-3}$$
$$\vdots$$
$$S_{n-1} \;=\; 1(1+i)^{[n-(n-1)]}$$
$$\;=\; 1(1+i)^1$$
$$S_n \;=\; 1(1+i)^{n-n}$$
$$\;=\; 1(1+i)^0$$
$$\;=\; 1.$$

The total amount of the annuity at the end of the term (after n payment intervals) is thus

$$S \;=\; S_1 + S_2 + \cdots + S_{n-1} + S_n$$
$$\;=\; (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^1 + 1. \tag{4.1}$$

This is a rather formidable-looking expression (which, by the way, is an example of the sum of a typical so-called "geometric progression"). Surprisingly, the sum is quite easily calculated using a trick (if you are not sure about the manipulations, choose a specific value for n (say $n = 4$) and i (say $i = 0{,}1$) and check in detail).

Multiply the left and right side of the expression by $(1+i)$. This gives

$$(1+i)S = (1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^2 + (1+i) \tag{4.2}$$

(since, eg, $(1+i)(1+i)^{n-4} = (1+i)^{n-3}$).

If we now subtract expression (4.1) from expression (4.2), we obtain

$$(1+i)S - S = (1+i)^n - 1$$

since all other powers of $(1+i)$ on the right-hand side become zero
$((1+i)^k - (1+i)^k = 0)$.

$$S + iS - S \;=\; (1+i)^n - 1$$
$$iS \;=\; (1+i)^n - 1$$
$$S \;=\; \frac{(1+i)^n - 1}{i}.$$

This formula is also known as the future value of an annuity and is denoted by $s_{\overline{n}|i}$. This is expressed in words as "s angle n at i". **The $\overline{}\rceil$ "angle" is only a symbol that we use when we want to indicate that the future value formula must be used. There is no $\overline{}\rceil$ symbol on any calculator.**

This calculation was done for a periodic payment of R1 per period. If the payment is R1 per period, then we simply have to multiply $s_{\overline{n}|i}$ by R.

Definition 4.2 *If the payment in rand, made at each payment interval in respect of an ordinary annuity certain at interest rate i per payment interval, is R, then the amount or future value of the annuity after n intervals is*

$$S = R s_{\overline{n}|i} = R\left(\frac{(1+i)^n - 1}{i}\right).$$

Note

1. All standard texts on annuities will, at this stage, refer you to pages of tables for $s_{\overline{n}|i}$. Fortunately for you, because of your calculator, they are not needed, since you can, in the wink of an eye, calculate

$$S = R\left(\frac{(1+i)^n - 1}{i}\right)$$

   for any values of i and n.

2. Confusion exists regarding the symbols used for the amount or future value and the payment, so be careful to check the particular usage in any text you consult. Fortunately, the notation $s_{\overline{n}|i}$ is fairly widely accepted (sometimes with r being used instead of i).

3. The annual interest rate j_m compounded m times per year $\frac{j_m}{m}$ is denoted by i while the number of interest compounding periods tm is denoted by n.

Exercise 4.2

1. Use the formula to confirm the results of exercise 4.1.

2. Mrs Dooley decides to save for her daughter's higher education and, every year, from the child's first birthday onwards, puts away R1 200. If she receives 11% interest annually, what will the amount be after her daughter's 18th birthday?

3. What is the accumulated amount (future value) of an annuity with a payment of R600 four times per year and an interest rate of 13% per annum compounded quarterly at the end of a term of five years?

Next we look at the **present value of an annuity**. As stated in the preceding section, this is the sum of the present values of all payments.

Definition 4.3 *The present value of an annuity is the amount of money that must be invested now, at i percent, so that n equal periodic payments may be withdrawn without any money being left over at the end of the term of n periods.*

Again, an example should clarify the concepts involved.

Example 4.5

Calculate the present value of an annuity that provides R1 000 per year for five years if the interest rate is 12,5% per annum.

The present value of the first payment is P_1:

$$\begin{aligned} P_1 &= 1\,000(1+0,125)^{-1} \\ &= 888,89 \end{aligned}$$

Remember this means that R888,89 invested now at 12,5% will yield R1 000 in one year.

Similarly, the present value of the four other payments is:

$$
\begin{aligned}
P_2 &= 1\,000(1+0{,}125)^{-2} \\
&= 790{,}12 \\
P_3 &= 1\,000(1+0{,}125)^{-3} \\
&= 702{,}33 \\
P_4 &= 1\,000(1+0{,}125)^{-4} \\
&= 624{,}30 \\
P_5 &= 1\,000(1+0{,}125)^{-5} \\
&= 554{,}93
\end{aligned}
$$

The present value of the annuity is:

$$
\begin{aligned}
P &= P_1 + P_2 + P_3 + P_4 + P_5 \\
&= 1\,000(1{,}125)^{-1} + 1\,000(1{,}125)^{-2} + 1\,000(1{,}125)^{-3} \\
&\quad + 1\,000(1{,}125)^{-4} + 1\,000(1{,}125)^{-5} \\
&= 3\,560{,}57
\end{aligned}
$$

Thus R3 560,57 must be invested now at 12,5% to provide for five payments of R1 000 each at yearly intervals commencing one year from today.

The formula for calculating the present values could be derived in the same way as was done above for the accumulated amount or future value. However, it is far quicker if we recognise that the present value of the annuity is the present value of the amount S that has accumulated by the end of the nth period. We can simply state that

$$
P = \frac{S}{(1+i)^n}
$$

but

$$
S = \frac{(1+i)^n - 1}{i}
$$

so

$$
P = \frac{(1+i)^n - 1}{i(1+i)^n}.
$$

This formula is also known as the present value of an annuity. It is denoted by $a_{\overline{n}|i}$ and in words is expressed as "a angle n at i". **The symbol ⌐ (angle) is only used when we want to indicate that the present value formula must be used.**

This calculation was performed for a periodic payment of R1. If the payment is R per period, the following applies:

Definition 4.4 *If the payment in rand made at each payment interval for an ordinary annuity certain, at interest rate i per payment interval, is R for a total of n payments, then the present value is*

$$
\begin{aligned}
P &= R a_{\overline{n}|i} \\
&= R\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right).
\end{aligned}
$$

Note

The comments made above with regard to notation and the use of tables in the case of the amount also apply to the present value. The annual interest rate j_m compounded m times per year $\frac{j_m}{m}$ is denoted by i, while the number of interest compounding periods tm is denoted by n.

Exercise 4.3

1. Use the formula to check the result of example 4.5 above.

2. Max puts R3 000 down on a second-hand car and contracts to pay the balance in 24 monthly instalments of R400 each. If interest is charged at a rate of 24% per annum, payable monthly, how much did the car originally cost when Max purchased it? How much interest does he pay?

3. Determine the present value of an annuity with semi-annual payments of R800 at 16% per year compounded half-yearly and with a term of ten years.

NB

Note that $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ are very simply related.

To be specific (check!)

$$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}.$$

Thus $s_{\overline{n}|i}$ is the future value of $a_{\overline{n}|i}$, and $a_{\overline{n}|i}$ is the present value of $s_{\overline{n}|i}$.

Study unit 4.3

4.3   Annuities due

After working through this section you should be able to

> ▷ relate an annuity due to an annuity certain and use the relationship to determine the value an annuity due.

As stated in section 4.1, an annuity due is an annuity in respect of which payment falls due at the beginning of each payment interval. A typical example of an annuity due is the monthly rent for a house. Note that, whereas the first payment of an ordinary annuity is at the end of the first payment interval, that of an annuity due is at the beginning. Furthermore, whereas the last payment of an ordinary annuity is at the end of the term, that of an annuity due is one payment interval before the end of the term. These differences are illustrated by the following time lines:

Ordinary annuity

Annuity due
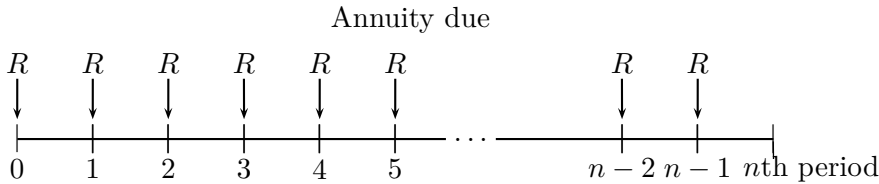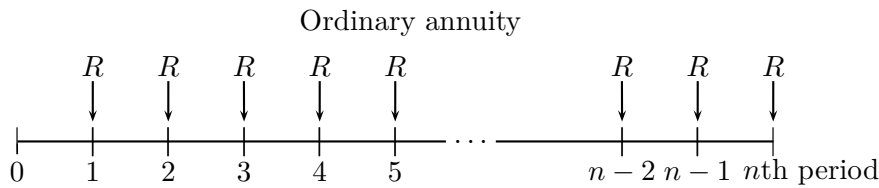
If we bear these differences with regard to the beginning and end of the two types of annuity in mind, no new formulæ are required. We may thus use those in the previous section as is shown in the following example:

⌐**Example 4.6**

If the monthly rental on a building is R1 200 payable in advance, what is the equivalent yearly rental? Interest is charged at 12% per annum compounded monthly. How much interest is paid?

The time line is as follows:
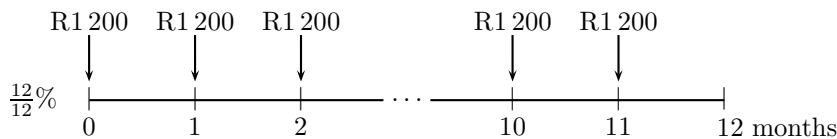
What the question in fact asks is: What rental must be paid as a lump sum at the beginning of the year instead of monthly payments? In other words, this is a present value-type calculation. You will recall, however, that, in the definition of $a_{\overline{n}|i}$ in the previous section, there was no payment at instant 0 but one at instant 12. Now there is one at the beginning but not at the end. This means that the first payment does not have to be discounted and that the remaining 11 payments form an ordinary annuity. Thus the equivalent yearly rental (ER) is given by

$$ER = \text{First payment} + \text{Present value of remaining 11 payments}$$

that is:

$$
\begin{aligned}
ER &= 1\,200 + 1\,200 a_{\overline{11}|0,12\div12} \\
&= 1\,200 + 12\,441{,}15 \\
&= 13\,641{,}15.
\end{aligned}
$$

The equivalent yearly rental is R13 641,15. The interest paid:

$$
\begin{aligned}
I &= 12 \times 1\,200 - 13\,641{,}15 \\
&= 14\,400 - 13\,641{,}15 \\
&= 758{,}85
\end{aligned}
$$

The interest paid is R758,85.   ⌐

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Exercise 4.4

What is the present value of an annuity due with quarterly payments of R500 at an interest rate of 16% per annum compounded quarterly and a term of six years?

Although, as I said, we do not, strictly speaking, need another formula, if you have to work a lot with annuities due, a formula is useful. After the last two exercises and the previous example, you should have no trouble deriving such a formula.

Exercise 4.5

Use the relationship established above between the present value of an annuity due and an ordinary annuity, namely

PV (of annuity due with n periods)

   =   first payment $+ PV($ of ordinary annuity with $(n-1)$ periods)

to show that the present value of an annuity due is given by

$$P = (1 + i)Ra_{\overline{n}|i}$$

and the *future value*, or *accumulated amount*, by

$$S = (1 + i)Rs_{\overline{n}|i}.$$

Thus

The present value of an annuity due is given by

$$P = (1 + i)Ra_{\overline{n}|i}$$

and the *future value*, or *accumulated amount*, by

$$S = (1 + i)Rs_{\overline{n}|i}.$$

Using your formula, the next exercise should be simple.

Exercise 4.6

John creates a fund (account) for his son's university education. At the *beginning* of each year, he deposits R3 000 in this account that earns 10% per year. Determine the balance in the fund at the end of year 12.

Note

In some textbooks you will find the symbols $\ddot{a}\,\overline{n}|i$ and $\ddot{s}\,\overline{n}|i$ being used. These two symbols are defined as

$$\ddot{a}\,\overline{n}|i = (1+i)a\,\overline{n}|i$$

and

$$\ddot{s}\,\overline{n}|i = (1+i)s\,\overline{n}|i.$$

4.4 Deferred annuities and perpetuities

After working through this section you should be able to

▷ describe deferred annuities and perpetuities so as to relate these to ordinary annuities and to use the relationship to value the former.

By now you have mastered all the important techniques in the calculation of annuities at different times. What remains are variations on the theme. In this section we shall consider two examples of these variations.

Definition 4.5 *A deferred annuity is an annuity where the first payment is made a number of payment intervals after the end of the first interest period.*

Essentially, however, this entails no additional complications. All you have to do is to work from a base date that is one period before the first payment so that ordinary annuities can be used and then discount to the actual beginning, as illustrated in the following example.

⌐**Example 4.7**

Jonathan intends to start a dry cleaning business and wishes to borrow money for this purpose. He feels that he will not be able to repay anything for the first three years but, thereafter, he is prepared to pay back R20 000 per year for five years. The bank agrees to advance him money at 18% interest per annum. How much would they be willing to advance him now under these conditions?

The time line is as follows:
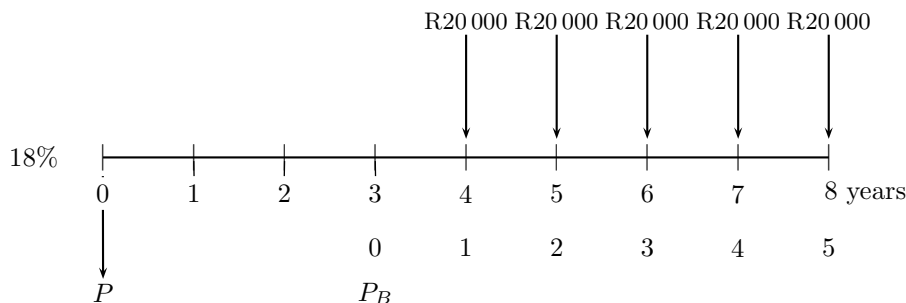
Taking year three as base date, the payments may be viewed as an ordinary annuity, with payments of R20 000 as stipulated.

$$P_B = 20\,000a_{\overline{5}|0,18}$$
$$= 62\,543{,}42$$

The present value on this base date is therefore R62 543,42. However, this is the value three years from now.

To obtain the present value P (today or now) we have to discount this amount back for three years by using the compound interest formula and the stipulated interest rate.

$$P = S(1 + \frac{j_m}{m})^{-tm}$$
$$= P_B(1 + 0{,}18)^{-3 \times 1}$$
$$= 62\,543{,}42 \times 1{,}18^{-3}$$
$$= 38\,065{,}86$$

Therefore, the bank will now advance him R38 065,86.

Another example of this type appears in the evaluation exercise.

The second type of variation now follows.

Definition 4.6 *A perpetuity is an annuity with payments that begin on a fixed date and continue forever. That is, it is an annuity that does not stop.*

Examples of perpetuities are scholarships paid from an endowment, the interest payments from an amount of money invested permanently and the dividends on a share, provided, of course, that the company does not cease to exist.

The time line representation of a perpetuity is as follows:
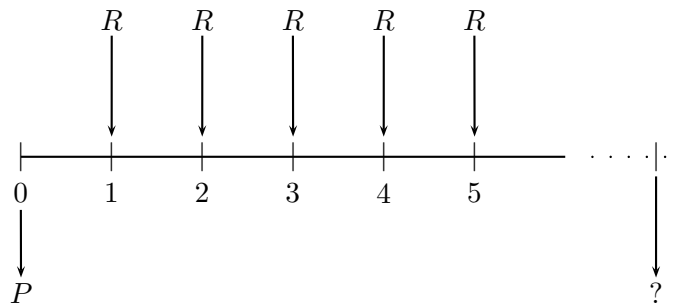
We cannot, of course, refer to the accumulated sum of a perpetuity, since the term of the perpetuity does not end. We can, however, calculate the present value of a perpetuity – it is simply the sum of the present values of the individual payments.

$$P = R(1 + i)^{-n}.$$

R is the payment, i is the interest rate and $n =$ the period.

We can then write the present value of the perpetuity, P, as

$$P = R(1 + i)^{-1} + R(1 + i)^{-2} + R(1 + i)^{-3} + \ldots \tag{4.3}$$

We now multiply both sides of the expression by $(1 + i)$. This gives

$$(1 + i)P = R + R(1 + i)^{-1} + R(1 + i)^{-2} + \dots \qquad (4.4)$$

If we subtract expression (4.3) from expression (4.4) we obtain

$$
\begin{aligned}
(1 + i)P - P &= R \\
P + iP - P &= R \\
iP &= R
\end{aligned}
$$

so that

$$P = \frac{R}{i}.$$

⌐**Example 4.8**

How much should a donor give Unisa to fund a scholarship of R10 000 per year indefinitely at an interest rate of 12,5% per year?

Now $R = 10\,000$ and $i = 0{,}125$.

$$
\begin{aligned}
P &= \frac{R}{i} \\
&= \frac{10\,000}{0{,}125} \\
&= 80\,000
\end{aligned}
$$

He should therefore donate an amount of R80 000. ⌐

4.5 General and increasing annuities

After working through this section you should be able to

▷ describe general and increasing annuities and relate them to ordinary annuities, and use the relationship to determine the value.

Study unit 4.5

You might think that, by now, we have exhausted all the possibilities with regard to annuities. This is not, however, the case. Up to now we have assumed that payment periods exactly match the interest periods (ie that payments are made at the beginning or end of each interest period). This need not be the case. Payments could be made more or less frequently than interest is compounded.

Definition 4.7 *A series of payments where the payment periods do not match the interest periods exactly is referred to as a general annuity.*

The best way of solving general annuity problems is to replace the specified interest rate and period with an equivalent interest rate that corresponds to the period of the payments. (You saw how to perform interest rate conversions of this kind in the previous chapter using the conversion of interest rate formula as an aid.) Alternatively, the payments may be replaced to coincide with the interest dates. The latter method is a little more complicated. Generally, because the first method is sufficient, we shall confine ourselves to an example using this method.

Example 4.9

In terms of an agreement with his creditors, Ivan makes payments of R1 200 into an account at the end of every two month period for five years. Determine the accumulated value of these payments if interest is compounded quarterly at 13,5% per annum.

The problem may be represented by the following time line:
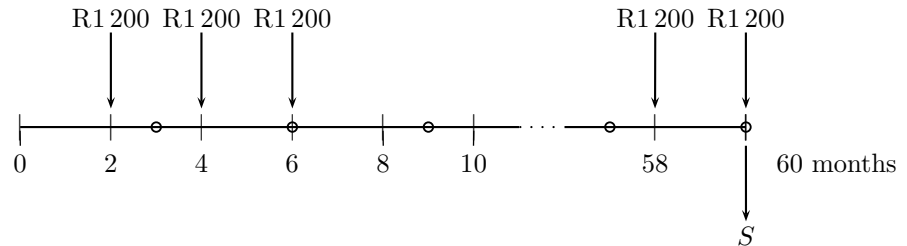
The points at which interest is credited are indicated by ∘.

We must convert the interest rate that is compounded quarterly to compounded every two months (bi-monthly).

Thus

$$
\begin{aligned}
j_n &= n\left(\left(1 + \tfrac{j_m}{m}\right)^{\frac{m}{n}} - 1\right) \\
\text{with} \quad n &= 6 \\
m &= 4 \\
j_m &= 0{,}135
\end{aligned}
$$

$$
\begin{aligned}
j_6 &= 6\left(\left(1 + \tfrac{0{,}135}{4}\right)^{\frac{4}{6}} - 1\right) \\
&= 0{,}13425 \\
&= 13{,}425\%.
\end{aligned}
$$

This is the nominal annual rate for equivalent bi-monthly compounding. But be *careful* – for the annuity calculation, we need to divide this by 6 to obtain the bi-monthly rate for compounding. Do this to obtain 2,2375299% per period.

In terms of a time line our problem is now
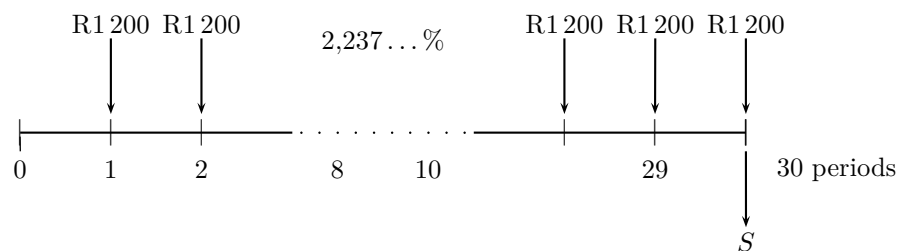
We have reduced it to an ordinary annuity with 30 payments at 2,237...% interest per period.

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$$S = 1\,200s_{\overline{5 \times 6}|0.13425\ldots \div 6}$$
$$= 50\,534{,}44$$

The sum accumulated in five years is thus R50 534,44.

You may test your skills on another general annuity in the evaluation exercise.

Another variation in the payment structure of an annuity that we should consider is when the payments increase in each payment period.

Definition 4.8 *An increasing annuity is a series of payments that increases by a constant amount with every payment.*

Suppose an annuity is payable annually for n years. The first payment is R, and the amount of each subsequent payment increases by Q. The interest rate per year is i.

The time line is as follows:
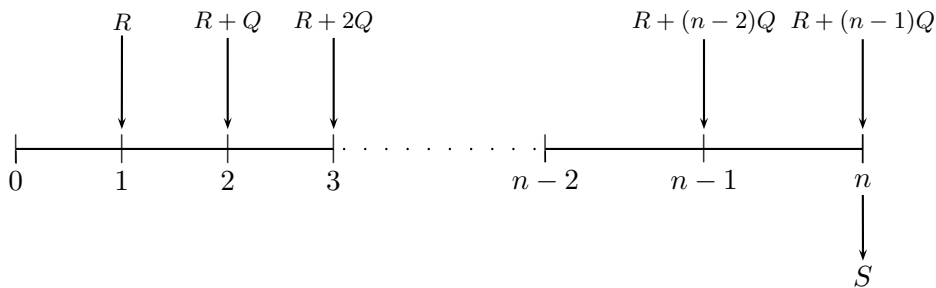
We may consider this time line as the combination of n annuities:
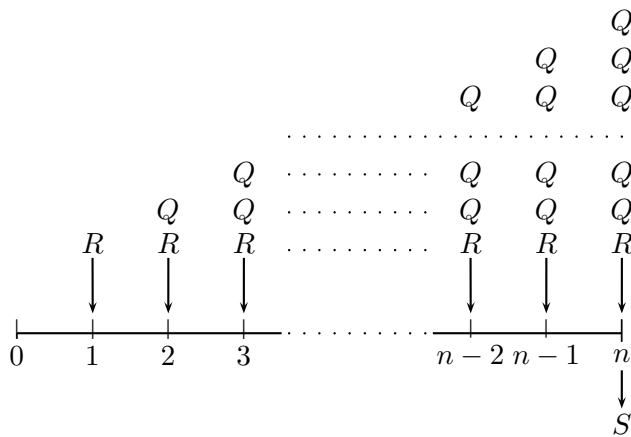
The future value of these n annuities is:

$$
\begin{aligned}
S = \; & R((1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1) \\
& + Q((1+i)^{n-2} + (1+i)^{n-3} + \cdots + (1+i) + 1) \\
& + Q((1+i)^{n-3} + \cdots + (1+i) + 1) \\
& + \ldots \\
& + Q((1+i)^{2} + (1+i) + 1) \\
& + Q((1+i) + 1) \\
& + Q.
\end{aligned}
\tag{4.5}
$$

We use exactly the same trick as before and multiply the equation on both sides by $(1+i)$ to get

$$
\begin{aligned}
(1+i)S \;=\; & R((1+i)^n + (1+i)^{n-1} + \cdots + (1+i)^2 + (i+1)) \\
& + Q((1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^2 + (i+1)) \\
& + Q((1+i)^{n-2} + \cdots + (1+i)^2 + (1+i)) \\
& + \ldots \\
& + Q((1+i)^3 + (1+i)^2 + (1+i)) \\
& + Q((1+i)^2 + (1+i)) \\
& + Q(1+i).
\end{aligned}
\tag{4.6}
$$

Subtracting (4.5) from (4.6) gives:

$$
\begin{aligned}
(1+i)S - S \;=\; & R((1+i)^n - 1) \\
& + Q((1+i)^{n-1} - 1) \\
& + Q((1+i)^{n-2} - 1) \\
& + \ldots \\
& + Q((1+i)^3 - 1) \\
& + Q((1+i)^2 - 1) \\
& + Q((1+i) - 1) \\
\;=\; & R((1+i)^n - 1) \\
& + Q((1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^3 + (1+i)^2 + (1+i)) \\
& - (n-1)Q \\
\;=\; & R((1+i)^n - 1) \\
& + Q((1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^3 + (1+i)^2 + (1+i) + 1) \\
& - (n-1)Q - Q \\
\;=\; & R\left((1+i)^n - 1\right) + Qs_{\overline{n}|i} - nQ.
\end{aligned}
$$

$$
\begin{aligned}
S + iS - S \;&=\; R\left((1+i)^n - 1\right) + Qs_{\overline{n}|i} - nQ \\
iS \;&=\; R\left((1+i)^n - 1\right) + Qs_{\overline{n}|i} - nQ \\
S \;&=\; R\left(\frac{(1+i)^n - 1}{i}\right) + \frac{Q}{i}s_{\overline{n}|i} - \frac{nQ}{i} \\
\;&=\; Rs_{\overline{n}|i} + \frac{Q}{i}s_{\overline{n}|i} - \frac{nQ}{i} \\
\;&=\; s_{\overline{n}|i}\left(R + \frac{Q}{i}\right) - \frac{nQ}{i}.
\end{aligned}
$$

⌈**Example 4.10**

> You take out an endowment policy that stipulates that the first payment of
> R1 200 is due in one year and, thereafter, the payment is increased by R200 at
> the end of every year. This policy matures in 20 years and the expected interest
> rate per year is 15%. What amount could you expect to receive after 20 years?
>
> This is an increasing annuity with $R = 1\,200$, $Q = 200$, $n = 20$ and $i = 0{,}15$.

$$S \;=\; s_{\overline{n}|i}\left(R + \frac{Q}{i}\right) - \frac{nQ}{i}$$

$$S \;=\; \left(1\,200 + \frac{200}{0{,}15}\right) s_{\overline{20}|0{,}15} - \frac{20 \times 200}{0{,}15}$$

$$\;=\; 259\,523{,}74 - 26\,666{,}67$$

$$\;=\; 232\,857{,}07$$

The expected future value of the policy is R232 857,07.

Exercise 4.7

In the above example, what is the total amount of interest that you will receive?

(Hint: $1 + 2 + 3 + \cdots + n = \frac{n(n-1)}{2}$ where n is the number of periods.)

4.6   Summary of Chapter 4

The basic concepts pertaining to annuities and their uses were defined, described and illustrated in this chapter.

An *annuity* was defined as a sequence of equal payments at equal intervals of time.

The differences between *ordinary annuities* and *annuities due* were pointed out, namely that in respect of the former, payments are made at the same time that interest is credited, whereas in respect of the latter, payments are made at the beginning of an interval. If payments never cease, the annuity is known as a *perpetuity.*

The formula for the *amount* or *future value* (S) of an ordinary annuity certain was derived, namely

$$S \;=\; Rs_{\overline{n}|i}$$

$$\;=\; R\left(\frac{(1+i)^n - 1}{i}\right).$$

So, too, was that for the present value (P), namely

$$P \;=\; Ra_{\overline{n}|i}$$

$$\;=\; R\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right).$$

Note: $s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}$.

In these formulæ, R is the payment made at the end of each payment interval for n intervals at interest rate i per interval.

The corresponding formulæ for *annuities due* were derived, namely

$$S \;=\; (1+i)Rs_{\overline{n}|i}$$

$$P \;=\; (1+i)Ra_{\overline{n}|i}.$$

The application of these formulæ to various types of annuity was considered, including the extension to *deferred* and *general annuities*.

A formula for the present value of a perpetuity was derived, namely

$$P = \frac{R}{i}.$$

We also indicated how the future value of an increasing annuity can be calculated and the formula was derived, namely.

$$S = \left(R + \frac{Q}{i} \right) s_{\overline{n}|i} - \frac{nQ}{i}.$$

4.7 Evaluation exercises

1. Determine the future and present value of an ordinary annuity with payments of R200 per month for five years at 18% per annum compounded monthly. What is the total interest received?

2. Suppose the annuity just described is not an ordinary annuity, but an annuity due. What would the future and the present value be then, and what would be the interest paid?

3. An ordinary annuity with payments of R400 per quarter and a term of six years, at an interest rate of 20% per annum compounded quarterly, is deferred for two years. In other words, the first payment will be made at the end of the first quarter two years hence and the last payment will be made eight years hence. What is the present value?

4. A contract requires payments of R25 000 semi-annually at the end of every six month period for five years, plus a final additional payment of R100 000 at the end of the five years. What is the present value of the contract at the date of commencement if money is worth 15% per annum compounded monthly?

5. Suppose an annuity is payable annually for n years. The first payment is R and thereafter each subsequent payment increases by $r\%$. The interest rate per year is i. Derive a formula for the future value of such an annuity.

   (Hint: Multiply S by $\frac{(1+i)}{(1+r)}$.)

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Amortisation and sinking funds

Outcome of chapter

> To be able to calculate the repayment of loans by means of both amortisation and sinking funds.

Key concepts

- ✓ Amortisation
- ✓ Amortisation schedule
- ✓ Buyer's equity

- ✓ Seller's equity
- ✓ Real cost
- ✓ Sinking fund

5.1 Amortisation

> *After working through this section you should be able to*
>
> ▷ explain the concepts of amortisation and buyer's and seller's equity and be able to calculate the repayment on a mortgage loan;
>
> ▷ set up an amortisation schedule for a loan;
>
> ▷ reschedule payments on a loan for changes in interest rate or term;
>
> ▷ calculate the real cost of a loan for a given assumed rate of inflation.

Let us start this section with a definition and an example.

Definition 5.1 *A loan is said to be amortised when all liabilities (that is both principal and interest) are paid by a sequence of equal payments made at equal intervals of time.*

⌐**Example 5.1**

A loan of R10 000 with interest of 14% compounded quarterly is to be amortised by equal quarterly payments over five years. The first payment is due at the end of the first quarter. Calculate the payment.

The time line is as follows:

R10 000

$$\frac{14}{4}\%$$
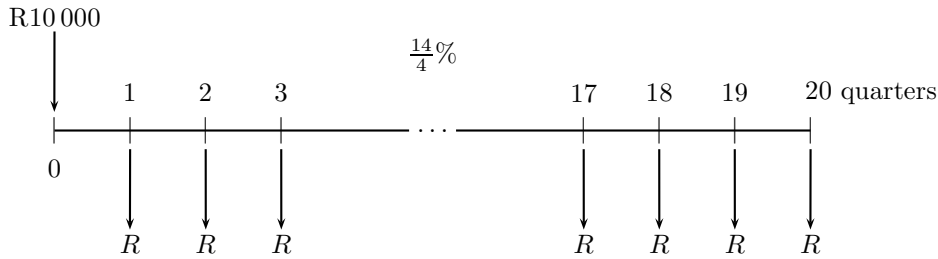
It is evident that the 20 payments form an ordinary annuity with a present value of
R10 000 and an interest rate of $14\% \div 4$.

$$
\begin{aligned}
P &= Ra_{\overline{n}|i} \\
R &= \frac{10\,000}{a_{\overline{5\times4}|0,14\div4}} \\
&= 703{,}61
\end{aligned}
$$

Thus the quarterly payment is R703,61.

This example illustrates a typical amortisation problem. A loan of present value P
rand must be amortised over n payments at interest rate i per payment interval. What
is the amount of the payment R? As pointed out in the example, the n payments
form an ordinary annuity; hence we may use the formula for the present value of such
an annuity to calculate R, namely

$$
\begin{aligned}
P &= Ra_{\overline{n}|i} \\
\text{or} \quad R &= \frac{P}{a_{\overline{n}|i}}.
\end{aligned}
$$

Definition 5.2 *If a loan of present value P rand must be amortised over n payments
at interest rate i per payment interval, then the amount of the payment R is*

$$
R = \frac{P}{a_{\overline{n}|i}}.
$$

Exercise 5.1

1. You purchase a small apartment for R180 000 with a down payment (often
   referred to as a deposit) of R45 000. You secure a mortgage bond with a bank
   for the balance at 18% per annum compounded monthly, with a term of 20
   years. What are the monthly payments?

2. Your Great Aunt Agatha dies and leaves you an inheritance of R60 000 which is to be paid to you in 10 equal payments at the end of each year for the next 10 years. If the money is invested at 12% per annum, how much do you receive each year?

Let us pause for a moment to think about the mechanics of amortisation. Initially, the total amount loaned (ie the present value at time zero) is owed. However, as payments are made, the *outstanding principal*, or *outstanding liability* as it is also known, decreases until it is eventually zero at the end of the term. At the end of each payment interval, the interest on the outstanding principal is first calculated. The payment R is then first used to pay the interest due. The balance of the payment is thereafter used to reduce the outstanding principal. (If, for some reason or another, eg by default, no payment is made, the interest owed is added to the outstanding principal and the outstanding debt then increases. This will usually be accompanied by a rather strongly worded letter of warning to the debtor!) Since the outstanding principal decreases with time, the interest owed at the end of each period also decreases with time. This means that the fraction of the payment that is available for reducing the principal increases with time.

Note that, at any stage of the term, the amount outstanding just after a payment has been made is the present value of all payments that still have to be made.

The above concepts are all embodied and illustrated in the *amortisation schedule*, that is, a table indicating the distribution of each payment with regards to interest and principal reduction. This is illustrated in the following example.

⌐**Example 5.2**

Draw up an amortisation schedule for a loan of R5 000 that is repaid in annual payments over five years at an interest rate of 15% per annum.

$$
\begin{aligned}
R &= \frac{P}{a_{\overline{n}|i}} \\
&= \frac{5\,000}{a_{\overline{5}|0,15}} \\
&= 1\,491{,}58
\end{aligned}
$$

The payments are R1 491,58.

Year	Outstanding principal at year beginning[c]	Interest due at year end [a]	Payment	Principal repaid[b]	Principal at year end
1	5 000,00	750,00	1 491,58	741,58	4 258,42
2	4 258,42	638,76	1 491,58	852,82	3 405,60
3	3 405,60	510,84	1 491,58	980,74	2 424,86
4	2 424,86	363,73	1 491,58	1 127,85	1 297,01
5	1 297,01	194,55	1 491,58	1 297,03	0,00
Total		2 457,88	7 457,90	5 000,02[d]	

Note:

(a) The interest due at the end of year one is:

$$5\,000\left(1 + \frac{0{,}15}{1}\right)^1 - 5\,000 = 5\,750 - 5\,000$$
$$= 750$$

The interest due is R750.

For the third year, the amount that must be paid as interest is:

$$3\,405{,}60\left(1 + \frac{0{,}15}{1}\right)^1 - 3\,405{,}60 = 3\,916{,}44 - 3\,405{,}60$$
$$= 510{,}84$$

Interest paid is R510,84.

(b) The principal repaid is the difference between the payment and the interest due. (For example at the end of the first year, the principal repaid is R741,58 $(1\,491{,}58 - 750)$.)

(c) Outstanding principal at year beginning is equal to the previous principal at year beginning minus principal repaid. (For example, at the beginning of the second year, the outstanding principal is R4 258,42 $(5\,000 - 741{,}58)$.)

(d) Also note that, owing to rounding errors, the total principal repaid is in error by two cents.

Exercise 5.2

1. Draw up an amortisation schedule for a loan of R4 000 for three years at 12% per annum compounded half-yearly and repayable in six half-yearly payments.

2. Draw up an amortisation schedule for the first six payments of the loan in exercise 5.1(1), and one for the last six payments (ie the 235th up to the 240th payments).

As stated above and as well illustrated in the above amortisation schedule, the outstanding principal decreases over the term from the amount initially borrowed to zero at the end. In the case where the loan is used to buy property, this means that, as the amount owed to the lender decreases, the fraction of the property that the buyer has "paid off" increases until, at the end of the term, the buyer owes nothing on his property, which is then paid off in full. At any stage during the term, the part of the price of the property that the buyer has paid off is called the *buyer's equity*, and the part of the price of the property that remains to be paid is called the *seller's equity*. Obviously, by definition, we have the following relationship between the two:

Selling price = Buyer's equity + Seller's equity.   NB

Since the definition is quite clear, you should manage the following exercise without a prior example.

Exercise 5.3

With reference to the apartment that you bought in exercise 5.1(1) above, what is your equity in the house after 12 years?

Next we look at a slightly more complex, but fairly typical, type of problem that we have to analyse when deciding whether or not to buy a new house.

Example 5.3

Edgar, a dynamic young executive, calculates that he can sell his house so as to have R680 000 available for a down payment on a new house. The price that the seller is willing to accept for Edgar's dream house is R1 500 000. To this, Edgar will have to add an extra R152 000 made up of estate agent's commission, transfer fees and the premium on an insurance policy that will cover the outstanding principal owed in the event of Edgar's death. His company will pay him a housing subsidy of R2 500 per month and also has sufficient financial leverage to secure him the necessary mortgage bond at 12,5% per annum (compounded monthly) for a period of 20 years. Assuming that Edgar is, for the next few years, willing to commit himself to up to a third of his gross monthly salary of R30 000, should he buy or not?

He will have to borrow a total of R972 000 ($1\,500\,000 + 152\,000 - 680\,000$). At 12,5% per annum, over a period of 20 years, his monthly payment will be

$$R \;=\; \frac{972\,000}{a_{\overline{20\times12}|0,125\div12}}$$

$$=\; 11\,043{,}29.$$

From this, we must deduct his subsidy of R2 500, which means that Edgar must contribute R8 543,29 ($11\,043{,}29 - 2\,500$). Since this is less than a third of his gross monthly salary ($30\,000 \div 3 = 10\,000$), we would advise Edgar to buy.

You will find a similar example in the evaluation exercise.

You may well argue that it is all very well doing such calculations, but everyone knows that, nowadays, interest rates on mortgage bonds do not hold for very long. So what is the point of talking about equal payments and, if the interest rate does change, how can we handle it?

Yes, it is true that interest rates do change from time to time. But this implies that the payments change accordingly, as you may possibly have experienced to your irritation if you have a house with a mortgage bond on it. To recalculate the payments, we simply work from the present value, which is the outstanding principal on the date from which the change is implemented. It is usually assumed that the number of payments still due remains the same, although, in some cases, the term of the loan may be extended. The following example illustrates a calculation of this type:

Example 5.4

Jonathan purchases an apartment by making a down payment of R160 000, and obtains a 20-year loan for the balance of R480 000 at 13,5% per annum compounded monthly. After four-and-a-half-years, the bank adjusts the interest rate to 14,5% per annum compounded monthly. What is the new monthly amount that he must pay if the term of the loan remains the same?

Initially, the interest rate was 13,5% ÷ 12 per month and the number of payments to be made was 20 × 12.

$$R \;=\; \frac{480\,000}{a_{\overline{20 \times 12}|\,0,135 \div 12}}$$

$$=\; 5\,795,40$$

Initially his payments were R5 795,40. After four-and-a-half-years, the present value of the loan is

$$P \;=\; 5\,795,40\,a_{\overline{(20-4,5) \times 12}|\,0,135 \div 12}$$
$$=\; 450\,841,86$$

But the outstanding principal of R450 841,86 must be amortised over $15\frac{1}{2}$ years at an interest rate of 14,5% per annum compounded monthly.

$$R \;=\; \frac{450\,841,86}{a_{\overline{(20-4,5) \times 12}|\,0,145 \div 12}}$$

$$=\; 6\,101,07$$

The new payment is R6 101,07.

The evaluation exercise contains a similar example.   ⌐

When you consider all the preceding calculations for home loans, it is astounding what you really pay for a home! In example 5.4 above, we saw that after four-and-a-half-years and 54 payments, that is, after Jonathan had parted with R312 951,52 (54 × 5 795,40), he had paid off a mere R29 158,14 (480 000 − 450 841,86) of the amount he originally borrowed.

If we assume for now that the interest rate of 13,5% will remain fixed over the 20-year term of the loan, Jonathan will, over the period, pay back a total amount of R1 390 896 (5 795,40 × 240) to redeem his loan. This is nearly three times more than the amount originally borrowed. The total amount of interest he will pay is R910 896 (1 390 896 − 480 000). This is enough to discourage any prospective home owner, or to cause anyone who has already incurred such a debt to doubt his senses.

However, you can take comfort in the fact that these figures look worse than they really are. To put them into perspective, we must consider the effect of inflation. Inflation has the effect that the purchasing power of money today is much more than it would be in, say, ten or 20 years from now. (Just think of how many sweets R10 bought you 20 years ago, and how little you get for it today!) This is also the case with a loan. The R1 you borrow today is worth much more than the R1 you would pay back in ten or 20 years. We can therefore not simply add payments to be made over a considerable period of time and compare the result to an amount received today.

All amounts must be discounted to the same base date. (Remember the equating principle in Chapter 3?) We do this as follows: Suppose the expected rate of inflation is r per period over the term of a loan with n payments of R rand each. Then the total value of the n payments in terms of the rand of today (taking inflation into account) is

$$T_\nu = Ra_{\overline{n}|r}.$$

T_r, the total real cost of the loan is the difference between the total value (T_ν) of the loan and the actual principal borrowed:

$$T_r = T_\nu - P$$

This is simply the present value of the series of payments discounted at the inflation rate.

T_r is the total real cost of the loan, and we should compare this with the actual principal borrowed. This is illustrated by the following example:

Example 5.5

Suppose the interest rate on Jonathan's loan of R480 000 (example 5.4) will remain fixed at 13,5% for the full term of the loan. What is the total real cost of the loan if the expected average rate of inflation over the term of the loan is: 5,5% per year.

$$
\begin{aligned}
P &= Ra_{\overline{n}|i} \\
480\,000 &= Ra_{\overline{20\times12}|0,135\div12} \\
R &= 5\,795{,}40.
\end{aligned}
$$

His monthly payments are R5 795,40.

The total value of the loan is determine as follows:

$$
\begin{aligned}
T_\nu &= Ra_{\overline{n}|r} \\
&= 5\,795{,}40a_{\overline{20\times12}|0,055\div12} \\
&= 842\,492{,}65.
\end{aligned}
$$

The total value of his loan in today's money, considering inflation is R842 492,65.

In terms of today's money the amount that must be paid back, over and above the loan (R480 000), is the total value of the loan minus the original loan and that is the total real cost (T_r) of the loan.

$$
\begin{aligned}
T_r &= T_\nu - P \\
&= 842\,492{,}65 - 480\,000 \\
&= 362\,492{,}65
\end{aligned}
$$

The total real cost of the loan is R362 492,65.

5.2   Sinking funds

After working through this section you should be able to

   ▷ explain the sinking fund and the replacement fund concept and be able to apply it to debt servicing and investment problems;

A not infrequent method of discharging a debt is when the debtor agrees to pay to the creditor any interest due on the loan at the end of each payment interval, and the full principal borrowed (the so-called "face value of the debt") at the end of the term. In such cases, there is usually an agreement that the debtor will deposit enough money each period into a separate fund so that, just after the last deposit, which coincides with the end of the term, the fund amounts to the original debt incurred. This fund usually earns interest, but generally not at the same rate as the interest on the debt. The fund created in this manner is known as a *sinking fund*. Companies often use sinking funds to accumulate capital that is to be used later for the purchase of new fixed assets.

Definition 5.3 *A sinking fund is nothing other than a savings account used to accumulate the capital needed to pay back the principal value of a loan at the end of the term of the loan. If it is created for the replacement of, say, a machine, it is known as a replacement fund.*

The concepts are illustrated in the following example.

⌐**Example 5.6**

A debt of R10 000 bearing interest of 14% per annum, to be paid half-yearly, must be discharged by the sinking fund method. If the sinking fund earns interest at a rate of 12% per annum compounded quarterly, and the debt is to be discharged after five years, determine the size of the quarterly deposits. What is the total yearly cost of the debt?

Now, if the payments into the sinking fund are R rand each quarter, they must accumulate, together with the interest earned at 12% per annum, to the amount of R10 000 after five years. These payments constitute an ordinary annuity:

$$
\begin{aligned}
S &= R s_{\overline{n}|\,i} \\
10\,000 &= R s_{\overline{5 \times 4}|\,0,12 \div 4} \\
R &= 372{,}16
\end{aligned}
$$

The quarterly deposits are R372,16.

The total yearly cost of the debt consists of the four quarterly deposits plus the two half-yearly interest payments.

$$
\begin{aligned}
I &= Prt \\
&= 10\,000 \times 0{,}14 \times \frac{6}{12} \\
&= 700
\end{aligned}
$$

The half-yearly interest payment is R700.

$$
\begin{aligned}
\text{Total yearly cost } C &= \text{Interest} + \text{Sinking fund deposits} \\
&= 2 \times 700 + 4 \times 372{,}16 \\
&= 2\,888{,}64.
\end{aligned}
$$

Thus the total yearly cost (C) is R2 888,64.

Exercise 5.4

A debt of R20 000 will be discharged from a sinking fund, after six years. The debt interest is 16% per year, compounded quarterly. The sinking fund will earn interest at a rate of 12% per year, compounded semi-annually. Determine the semi-annual deposits in the fund. Determine the total annual cost of servicing the debt.

5.3   Summary of Chapter 5

Amortisation was defined as the process whereby a loan is paid off by a sequence of equal payments made at equal intervals of time. If a loan has an initial value of P, and must be amortised by means of n payments at interest rate i, then the payments are

$$
R = \frac{P}{a\,\overline{n}|i}.
$$

Several examples were considered, and the *amortisation schedule* was illustrated. In particular, the calculation of payments on a mortgage bond was described. The concept of equity was introduced and illustrated.

The concept of a sinking fund, whereby the interest on a loan is paid periodically and a deposit is paid into a fund so that the fund accumulates to the amount of the initial loan over the term of the loan, was described and illustrated. The payments into the sinking fund are

$$
R = \frac{S}{s\,\overline{n}|i}.
$$

5.4   Evaluation exercises

1. Mr and Mrs Newhouse purchase a duet for R475 000 and make a down payment of R350 000. They obtain a mortgage loan for the balance at 19% per annum, compounded monthly, and a term of 20 years. What are their monthly payments? After five years, what equity do they have in their duet? At that stage (ie after five years), their bank adjusts the interest rate to 18% per annum compounded monthly. What will their new monthly payments be if the term remains the same?

2. Draw up an amortisation table for a loan of R12 000 with interest at 10% per annum compounded annually and a term of five years.

3. You have R280 000 available for a deposit on a house that will cost you R960 000 (including transfer costs, etc) You can obtain a mortgage loan from the bank at 20% per annum compounded monthly and a term of 25 years. Your company is willing to grant you a housing subsidy of R2 000 per month. What would the monthly payments be in total, and what would they be after deduction of the subsidy? What is the total interest that will be paid over the term of the loan? How does this compare with the total real cost of the loan, assuming an inflation rate of 0,75% per month? And how does the total real cost of your share of the payments compare with the principal borrowed?

4. Mr Wheel N Deal wishes to borrow R50 000 for five years for a business venture. The Now Bank for Tomorrow is willing to lend him the money at 15% per annum if the debt is amortised by equal yearly payments. However, Yesteryear's Bank for Today will lend the money at 14% per annum, provided that a sinking fund is established with it, on which it will pay 11% per annum, to accumulate to the principal by the end of the term, with equal annual deposits. What is the difference in total annual payments between the two plans?

Evaluation of cash flows

Outcome of chapter

> To master the basic concepts of capital budgeting and the various methods that can be used to appraise investments that will yield cash flows at future dates.

Key concepts

- ✓ Capital budgeting
- ✓ Cash flows
- ✓ Payback method
- ✓ Average rate of return

- ✓ Internal rate of return
- ✓ Net present value method
- ✓ Profitability index
- ✓ Modified internal rate of return

6.1 Capital budgeting

> *After working through this section you should be able to*
>
> ▷ explain the capital budgeting process;
>
> ▷ compare two alternative investments with different cash flows using the payback and average rate of return mechanisms.

Capital budgeting involves long-term decision making on the use of funds. This implies the evaluation of investment opportunities, the essence of which is the detailed consideration of expected future cash flows. A *cash flow* may be defined as the receipt or expenditure of cash during an interval of time. For the sake of simplicity, it is generally assumed that cash flows occur at the end of each interval of time (usually at the end of each year).

Over the years, four basic methods of analysing cash flows have been developed in order to evaluate and choose between several investment proposals. They are the following:

1. The *payback method*, whereby the number of years' necessary to return the original investment is determined.

2. The *average rate of return method* (ARR), whereby the average annual income is compared with the average investment.

3. The *internal rate of return method* (IRR), which determines the interest rate that equates future returns to the present investment outlay.

4. The *net present value method* (NPV), which compares the present value of the future returns with the investment outlay.

The first two methods are the traditional approaches and are still commonly used, despite the fact that they are decidedly inferior to the last two. The last two methods belong to the class of so-called *discounted cash flow methods* and, as we shall see, are based on a proper consideration of the time value of money via compound interest calculations. The first two are only included for reference purposes and to establish certain basic concepts. For these reasons, they will just be discussed and illustrated briefly using the example below. The last two methods are the subject of the next two sections.

Finally, we question the assumption that the anticipated rate of return equals the borrowing rate, and introduce the modified internal rate of return to obviate this assumption.

Example 6.1

Use the payback method to evaluate the following two investment proposals, which require identical outlays of R10 000 each. The table lists the expected annual returns (ie cash inflows) for each proposal.

Year	Proposal A	Proposal B
	Investment of R10 000	Investment of R10 000
	Cash Inflows (R)	Cash Inflows (R)
1	3 000	1 000
2	3 000	1 000
3	4 000	1 000
4	5 000	7 000
5	6 000	12 000
6	7 000	16 000

It is evident from the table that, if proposal A is chosen, the initial investment will be paid back in three years, whereas, for proposal B, the "payback period" will be four years. Since, with the payback method, the principle of selection is based on which proposal has the shortest payback period, proposal A must be chosen.

The emphasis as regards the payback method is on the rapid recovery of capital outlay. Therein lies its two major weaknesses: First, it ignores cash flows beyond the payback period, as is evident from the above example where there are considerable

differences between the two proposals in years five and six; and secondly, it ignores the time value of money. For example, with the payback method, a proposal involving an investment of R10 000 and returning R5 000 in each of the first two years is equivalent to a second, also of R10 000, which returns only R2 000 in the first and R8 000 in the second year. In its favour it may be argued that this method is easy to understand, it focuses on a quick return and liquidity, which may be important and, in certain circumstances, it can give reasonable approximations to more sophisticated methods. Generally, however, these arguments are not sufficient to warrant its continued and widespread use, and you are referred to texts on financial management for a detailed consideration of the "pros and cons".

A method that overcomes the first shortcoming is the *average rate of return* (ARR) method. The ARR may be defined as the ratio of the average annual after-tax income to the level of investment. That is,

$$\text{Average rate of return (ARR)} = \frac{\text{Average after-tax income}}{\text{Investment level}}.$$

The investment level used varies in practice – it may, for example, be the initial outlay or it may be the average outlay. The next example illustrates this approach.

⌐ **Example 6.2**

A firm is considering two different investment possibilities, A and B, which are expected to yield the cash inflows tabled below for the stated level of investment. Use the average rate of return to choose between the two.

Year	Proposal A	Proposal B
	Investment of R20 000	Investment of R16 000
	After tax income (R)	After tax income (R)
1	2 000	1 000
2	2 000	1 000
3	2 000	1 000
4	2 000	1 500
5	2 000	1 500
6	2 000	2 000
7	2 000	2 500
8	2 000	3 500
Total	16 000	14 000

For proposal A, the average after-tax income is R2 000 (16 000 ÷ 8), whereas, for proposal B, it is R1 750 (14 000 ÷ 8). Comparing this with the initial outlay, namely R20 000 and R16 000 respectively, we find

$$\text{ARR(A)} = \frac{2\,000}{20\,000} = 10\%,$$

$$\text{ARR(B)} = \frac{1\,750}{16\,000} = 10{,}9\%.$$

On this basis, proposal B would be selected as the better one of the two.

Alternatively, both could be compared with the firm's cost of capital. If this was, say, 11,1%, then both might be rejected and other options investigated.   ⌐

As mentioned above, this method overcomes the first drawback. It is, however, evident that it too pays no consideration to the time value of money. Although, in the above example, proposal B is selected, the fact that proposal A has much higher early inflows than B is given no weight at all. The methods described in the following two sections are superior in this respect.

6.2 Internal rate of return

Study

unit

6.2

After working through this section you should be able to

▷ explain the internal rate of return concept;

▷ know the equation that must be formulated for the general problem of an investment with unequal cash flows at regular intervals;

▷ solve the equation to determine the internal rate of return in order to compare alternative investments.

As you saw in the previous examples, it often happens that the present value of an initial investment and the values of the subsequent cash flows are known.

Definition 6.1 *The compounded interest rate which, when used to discount the cash flows, will yield a present value equal to the initial investment, is known as the internal rate of return.*

The actual determination of the internal rate of return from the given cash flows and the determination of the present value of the investment are not always straightforward and, in most cases, we have to resort to numerical means, as you will see below. Nevertheless, armed with your calculator, there is no need for alarm.

We start with a simple example with which you are already familiar.

⌐**Example 6.3**

Trevor Sithole invests R10 000 in a project that promises to yield a single cash flow of R18 000 at the end of the fourth year. What is the anticipated annual (internal) rate of return?

In effect, what we are asking is what the interest rate i is such that the future value of R10 000 in four years will be R18 000. Expressed mathematically, we want to solve i so that

$$S = P(1 + i)^n$$

for a given S and P. That is, so that

$$18\,000 = 10\,000(1 + i)^4.$$

Or, equivalently, in present value terms, so that

$$P = S(1 + i)^{-n}.$$

By substituting we get

$$10\,000 = 18\,000(1 + i)^{-4}.$$

The time line is:
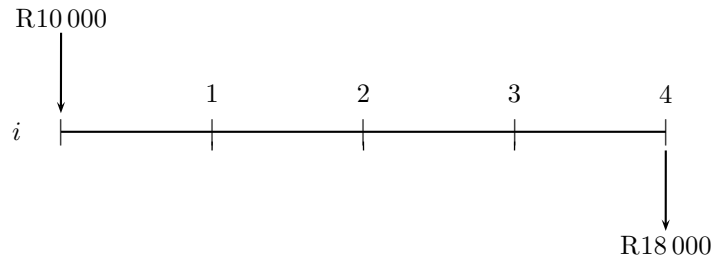
R10 000

We can easily solve for i by rearranging as follows:

$$
\begin{aligned}
(1+i)^4 &= \frac{18\,000}{10\,000} \\
&= 1{,}8
\end{aligned}
$$

$$
\begin{aligned}
(1+i) &= 1{,}8^{\frac{1}{4}} \\
&= 1{,}1583.
\end{aligned}
$$

$$
\begin{aligned}
i &= 1{,}1583 - 1 \\
&= 0{,}1583 \\
&= 15{,}83\% \text{ per annum}
\end{aligned}
$$

In other words, the internal rate of return is 15,83% per annum.

We can generalise this to derive a formula for i. Suppose an investment of present value I_{out} generates a single cash inflow of value C after n periods. Denote the internal rate of return by i. This is simply a compound interest calculation where the principle I_{out} accumulates to the amount C at an interest rate of i after n periods. Therefore

$$ C = I_{out}(1+i)^n $$

or, otherwise stated, the present value of the amount C at interest rate i is I_{out}. Therefore

$$ I_{out} = C(1+i)^{-n} $$

or

$$ (1+i)^n = \frac{C}{I_{out}}. $$

We wish to solve for i. This is done, as we saw above, by taking the nth root of both sides of the last equation, which gives

$$ 1 + i = \left(\frac{C}{I_{out}}\right)^{\frac{1}{n}}. $$

NB

$$i = \left(\frac{C}{I_{out}}\right)^{\frac{1}{n}} - 1.$$

This is the equation for the internal rate of return in the single payment case.

Note

When we have a formula that can be used to calculate the value of an unknown, as for example

$$i = \left(\frac{C}{I_{out}}\right)^{\frac{1}{n}} - 1$$

we say that we solve i analytically. On the other hand, a numerical method is one that, in some systematic way, tries different values for an unknown until it finds one that satisfies the given equation. The interval-halving algorithm is an example of such a numerical method.

Problems involving this equation are quite straightforward, and you should therefore manage the next exercise without any difficulty.

Exercise 6.1

If an investment of R10 000 returns a single flow of R16 500 in four years, what is the internal rate of return?

Next, we consider the case where an investment outlay, I_{out}, generates a uniform sequence of payments (cash flows), each of value C, for n intervals.
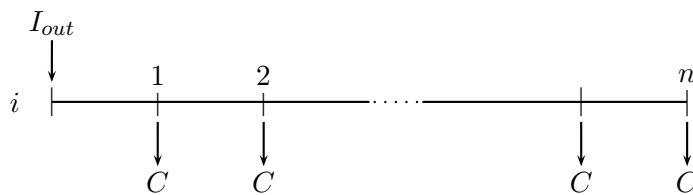
You will recognise that this series of payments forms an ordinary annuity with present value I_{out}, so that we can set

$$I_{out} = Ca_{\overline{n}|i} = C\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right).$$

Unfortunately this equation cannot be written as a formula for i in terms of I_{out}, C and n – as was the case for the single payment. However, we can solve the expression numerically.

To solve the above expression numerically, we do the following:

Divide by C on both sides and then subtract $\frac{I_{out}}{C}$ from both sides. This gives:

NB

$$f(i) = \left(\frac{(1+i)^n - 1}{i(1+i)^n}\right) - \frac{I_{out}}{C} = 0$$

This is the equation for the internal rate of return where an investment outlay, I_{out}, generates a uniform sequence of equal payments, each of value C for n intervals.

On the left-hand side we now have a function in i only, since the values of C, I_{out} and n are specified. Thus we write

$$f(i) = 0.$$

This is an equation with one unknown, namely i, which may be solved easily with the help of your calculator. This is illustrated in the following example:

Example 6.4

An investment with an initial outlay of R120 000 returns a constant cash flow of R24 000 per year for 10 years. What is the internal rate of return of the investment?

We now have

$$\frac{I_{out}}{C} = \frac{120\,000}{24\,000} = 5.$$

Thus

$$f(i) = \left(\frac{(1+i)^{10} - 1}{i(1+i)^{10}}\right) - 5 = 0.$$

Solving we find that

$$i = 0{,}151 \text{ and } f(i) = 0.$$

Thus to three decimal figures the internal rate of return is 15,100%.

OR

This investment may be treated as an ordinary annuity. The constant cash flow of R24 000 is the payment, while the initial outlay of R120 000 is the present value of the investment and the term is 10 years. We now have

$$P = Ra_{\overline{n}|i}$$

with

P	$=$	120 000
R	$=$	24 000
n	$=$	10
i	$=$	?.

Solving we find that $i = 0{,}151$. The internal rate of return is therefore 15,1%.

Exercise 6.2

An investment of R54 000 returns a cash flow of R9 500 for eight years. Calculate the internal rate of return.

Lastly, we consider the general case in which all m payments differ. We denote the sequence of payments by C_1, C_2, C_3, ..., C_m. That is, C_m is the cash flow at the end of the payment interval m. Examples are the values of the two proposals, A and B, in example 6.1 above. To be specific, using a time line for proposal A, the cash flows are as follows:
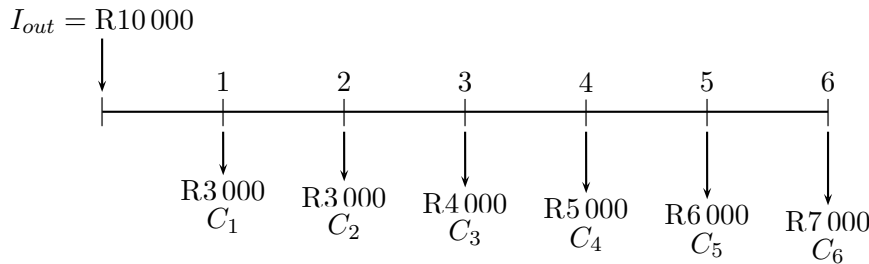
The present value (I_m) of the mth payment is then

$$I_m = \frac{C_m}{(1+i)^m}$$

where i is the internal rate of return.

Since the initial investment outlay (I_{out}) must equal the sum of all the present values, we have

$$I_{out} = \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \frac{C_3}{(1+i)^3} + \cdots + \frac{C_m}{(1+i)^m}.$$

This equation defines the internal rate of return, that is, just that value of i that satisfies the equation.

(Note that, if all $C_m = 0$ except, say, the last, then this reduces to the single payment case above.)

This equation appears complicated but once again, since I_{out}, m and all C_m are specified, it reduces to an equation of the form

$$f(i) = 0$$

with

$$f(i) = \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \frac{C_3}{(1+i)^3} + \cdots + \frac{C_m}{(1+i)^m} - I_{out}$$

that can be solved numerically (by means of the interval halving method). As long as m is not too large, we can handle this on our calculator. This case is illustrated in the next example.

Example 6.5

An investment of R120 000 generates three successive cash inflows of R60 000, R48 000 and R35 000 respectively. What is the internal rate of return?

It is a good idea to draw a time line for the cash flows:

$$I_{out} = R120\,000$$
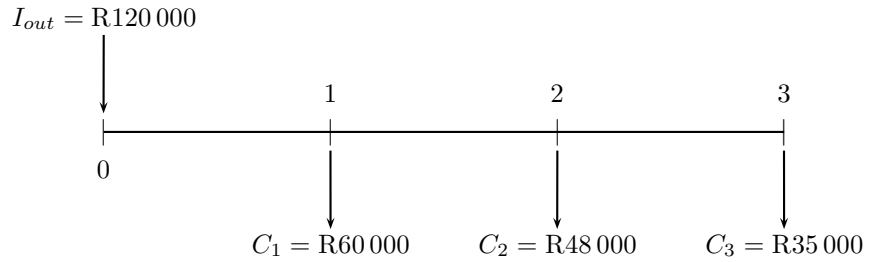
Now

$$f(i) = \frac{60\,000}{(1+i)} + \frac{48\,000}{(1+i)^2} + \frac{35\,000}{(1+i)^3} - 120\,000.$$

We must determine i such that $f(i) = 0$.

If we solve i by using our calculator, we will find that the internal rate of return for the investment is 10,3%.

Exercise 6.3

An investment outlay of R320 000 generates five successive yearly payments of R80 000, R110 000, R120 000, R90 000 and R70 000. What is the internal rate of return?

In practice, the internal rate of return can be used to choose between two alternative investment proposals, in which case the one with the highest internal rate of return is chosen. However, what is more relevant is the question whether the rate is sufficiently high enough to justify the commitment of funds to the venture. This means that the investor must compare the internal rate of return with the cost of acquiring capital. *If the internal rate of return exceeds the cost of capital, the proposal will usually be regarded as acceptable* (other things being equal, which, of course, may not be the case!).

For instance, in example 6.5, the investment of R120 000 returned 10,3% (IRR). If the cost of capital in this case is 9,5%, the project is acceptable, whereas if capital costs 11%, it is not.

Study
unit
6.3

6.3   Net present value

After working through this section you should be able to

▷ explain the net present value concept, know the steps required to determine it and be able to apply these steps in order to compare alternative investments;

▷ know and use the extension of net present value to the profitability index;

▷ relate and compare net present value, internal rate of return and profitability index for investment appraisal.

Another method that takes into account the time value of money when evaluating investment proposals is the *net present value method*. As you will see below, it is akin

to the internal rate of return method but is more direct and less laborious to apply. *It assumes that the cost of capital (K), that is, the rate at which one can borrow money over the life of a project, is known and then discounts all cash flows at this rate back to the present.* The net present value of all cash flows is then compared with the investment necessary to produce the inflows.

Definition 6.2 *Concisely, the net present value (NPV) of an investment proposal is the present value of all future cash inflows (PV_{in}) less the investment outlay (I_{out}):*

$$NPV = PV_{in} - I_{out}$$

If the NPV is positive (greater than zero), the proposal may be regarded as acceptable. If it is negative (smaller than zero), it is not. And if it is zero, the investor will be indifferent.

The following example illustrates the use of this method.

Example 6.6

A shopping complex may be purchased for R1 000 000. It is expected that it will return a uniform cash flow of R180 000 per year, in the form of rentals, for 10 years. If the cost of capital is 11,5% per annum, should the investor buy the complex? And what if the cost of capital is 12,5%?

The time line for the cash flows is as follows:
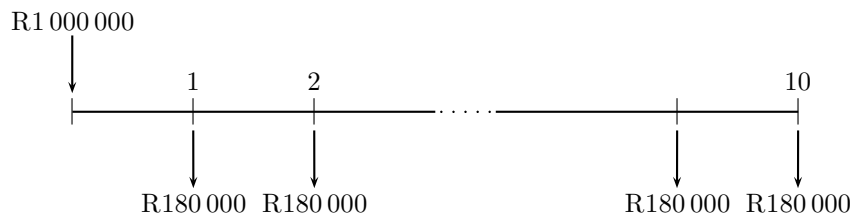
The series of rentals forms an annuity with 10 payments, so we can easily calculate the present value PV_{in} of the cash inflows:

If $K = 11{,}5\%$ then

$$
\begin{aligned}
PV_{in} &= 180\,000a_{\overline{10}|0,115} \\
&= 1\,038\,199
\end{aligned}
$$

The present value PV_{in} of the cash inflows is R1 038 199 (to the nearest rand).

$$
\begin{aligned}
NPV &= PV_{in} - I_{out} \\
&= 1\,038\,199 - 1\,000\,000 \\
&= 38\,199
\end{aligned}
$$

The net present value is R38 199.

If $K = 12{,}5\%$ then

$$
\begin{aligned}
PV_{in} &= 180\,000a_{\overline{10}|0,125} \\
&= 996\,558.
\end{aligned}
$$

The present value is R996 558 (to the nearest rand).

$$
\begin{aligned}
\text{NPV} &= 996\,558 - 1\,000\,000 \\
&= -3\,442
\end{aligned}
$$

The net present value is $-$R3 442. If the cost of capital is 11,5% the investor could purchase, whereas, if it is 12,5%, he should not.

It should be obvious to you that there is nothing in the method that restricts it to a uniform sequence of cash flows, as was used in the above example. If the sequence of cash flows consists of unequal payments C_m for $m = 1, 2, \ldots, n$, then we must discount each one separately. The net present value is then

$$
\text{NPV} = \frac{C_1}{1 + K} + \frac{C_2}{(1 + K)^2} + \cdots + \frac{C_n}{(1 + K)^n} - I_{out}.
$$

The next exercise illustrates this.

Exercise 6.4

An investment of R800 000 is expected to yield the following sequence of yearly cash flows over the next five years: R100 000, R160 000, R220 000, R280 000, R300 000. If the cost of capital is $K = 9\%$ per annum, should the investor proceed? And what if it is $K = 8\%$?

At this stage, a comment about the use of net present value for choosing between alternative proposals. *Using this method to choose between two proposals, both with a positive net present value, the rule is to choose the one with the highest positive net present value.* However, if they require investments of different sizes, this may not be the best choice. This is embodied in the concept of the *profitability index*, which is defined by

Definition 6.3

$$
\text{Profitability index } (PI) = \frac{\text{Present value of cash inflows}}{\text{Present value of cash outflows}}
$$

or

$$
\text{Profitability index } (PI) = \frac{NPV + \text{ outlays (initial investment)}}{\text{outlays (initial investment)}}.
$$

In cases where the only cash outflow is the initial investment outlay, this obviously simplifies the situation. You should be able to apply the index without difficulty in the next exercise.

Exercise 6.5

An investor contemplates two investment proposals, A and B. The investments and cash inflows are tabled below:

Year	Proposal A	Proposal B
	Investment of R100 000	Investment of R120 000
	Cash Inflow	Cash Inflow
1	85 000	90 000
2	65 000	90 000

Compare the use of net present value and the profitability index for choosing between two alternatives if the cost of capital is 12% per annum.

Finally, a few comments on the use of net present value (NPV) vis-à-vis the internal rate of return (IRR). Bear in mind that, in the general case, the net present value is given by

$$NPV = PV_{in} - I_{out}$$

with

$$
\begin{aligned}
PV_{in} &= \frac{C_1}{1+K} + \frac{C_2}{(1+K)^2} + \cdots + \frac{C_m}{(1+K)^m} \quad (K = \text{cost of capital}) \\
&= \sum_{n=1}^{m} \frac{C}{1+K} \qquad (\text{See Chapter 8 for the } \sum \text{ sign})
\end{aligned}
$$

or if only one cash flows is applicable then

$$PV_{in} = \frac{C}{1+K}$$

and with I_{out} **the initial investment.**

According to the definition, the internal rate of return (IRR) is the interest rate i that will make the present value of the cash inflows equal to the initial investment.

$$
\begin{aligned}
f(i) &= \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + \cdots + \frac{C_m}{(1+i)^m} - I_{out} = 0 \\
I_{out} &= \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + \cdots + \frac{C_m}{(1+i)^m} \\
&= \sum_{n=1}^{m} \frac{C_n}{(1+i)^n}
\end{aligned}
$$

or if only one cash flow is applicable

$$I_{out} = \frac{C}{1+i}.$$

Therefore

$$
\begin{aligned}
NPV &= PV_{in} - I_{out} \\
&= \frac{C}{1+K} - \frac{C}{1+i}.
\end{aligned}
$$

If a cash flow (C) of R100 is applicable and the cost of capital (say, 15%) is equal to the interest rate (say, 15%), then the NPV is equal to zero.

$$
\begin{aligned}
\text{NPV} \;=\;& \frac{C}{1+K} - \frac{C}{1+i} \\
=\;& \frac{100}{1+0{,}15} - \frac{100}{1+0{,}15} \\
=\;& 0
\end{aligned}
$$

This means that no profit or loss is made and the investor will be indifferent to investing in the project, since the cost of capital equals the internal rate of return.

If the cost of capital is larger (say, 18%) than the interest rate (say, 10%) and the cash flow is R100, then the NPV will be negative.

$$
\begin{aligned}
\text{NPV} \;=\;& \frac{C}{1+K} - \frac{C}{1+i} \\
=\;& \frac{100}{1+0{,}18} - \frac{100}{1+0{,}10} \\
=\;& 84{,}75 - 90{,}91 \\
=\;& -6{,}16 \\
<\;& 0
\end{aligned}
$$

If the NPV is negative, a loss is made and it would be unwise to invest in the project.

On the other hand, if the cost of capital is less (say, 15%) than the interest rate (say, 17%), then the NPV will be positive. (The cash flow is R100.)

$$
\begin{aligned}
\text{NPV} \;=\;& \frac{C}{1+K} - \frac{C}{1+i} \\
=\;& \frac{100}{1+0{,}15} - \frac{100}{1+0{,}17} \\
=\;& 86{,}96 - 85{,}47 \\
=\;& 1{,}49 \\
>\;& 0
\end{aligned}
$$

The NPV is positive, which means that a profit is made, so that it is advisable to invest in the project.

This may be extended to the case where there is more than one cash flow.

If we apply both methods to the same set of cash flows, then we see that the two methods are consistent with regard to the acceptance or rejection of alternative proposals.

The methods discussed in the previous two sections with their resulting decisions are summarised in the following table:

Internal rate of return (i) compared with cost of capital (K)	Net present value	PI	Decision on investment proposal
$i > K$	Positive	> 1	Accept
$i < K$	Negative	< 1	Reject
$i = K$	Zero	$= 1$	Indifferent

This summary must not be construed as implying that the three methods will result in the same decision when comparing two acceptable alternatives. In fact, there are examples, similar to exercise 6.5, where they produce conflicting results. There are many arguments about the relative merits of each of the three methods. In all our examples and exercises so far, we have considered only a single (initial) cash outlay. In practice, projects are more complicated. We shall consider this more general case in the next section.

6.4   Modified internal rate of return

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After working through this section you should be able to

▷ explain the modified rate of return concept;

▷ know and apply the steps that have to be followed in order to determine it;

▷ use it in order to evaluate an investment that involves multiple positive and negative cash flows.

</div>

Study

unit

6.4

As I stated above, so far we have considered only projects in which there is an initial cash outflow followed by a series of cash inflows. Schematically, this may be represented as follows:
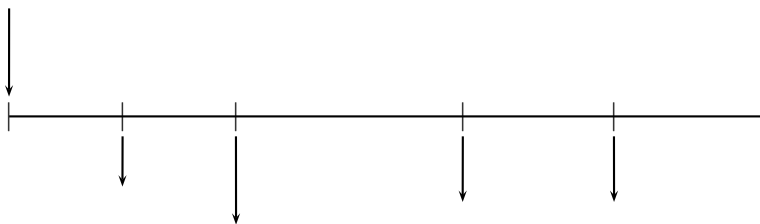
In practice, of course, many projects require initial outlays of cash for a number of years' before inflows commence. Typically:
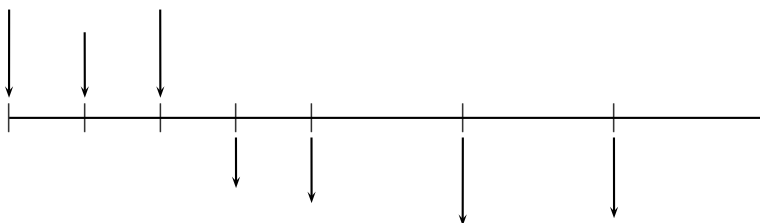
Indeed, in many cases, the cash outflows are not all confined to the initial few years. For example, it may be necessary to purchase new equipment or expand a factory at some stage during its life. For example:
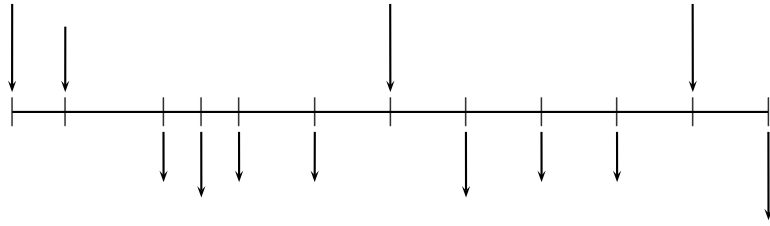
Now, in such cases, the question arises as to how the cash outflows should be discounted. The IRR and NPV both simply assume that this is at the same rate as cash inflows, namely at the internal rate of return, i, in the former case, and at the cost of capital in the latter case. The problem in both cases is that this is not realistic – money is not borrowed and invested at the same rate. Also, for the IRR, solving the equation $f(i) = 0$ may lead to ambiguities, since, in this case, more than one solution may exist. To avoid these difficulties, the so-called *modified internal rate of return (MIRR)* was introduced.

At the end of the lifetime of a project the present value of all the negative cash flows must be equal to the future value of the positive cash flows.

The internal rate of return that relates these two amounts, is known as the modified internal rate of return (MIRR).

$$
\begin{aligned}
PV_{out}\,(1+i)^n &= C \\
(1+i)^n &= \tfrac{C}{PV_{out}} \\
1+i &= \left(\tfrac{C}{PV_{out}}\right)^{\frac{1}{n}} \\
i &= \left(\tfrac{C}{PV_{out}}\right)^{\frac{1}{n}} - 1
\end{aligned}
$$

It is thus based on far more realistic assumptions than either the IRR or NPV methods.

NB

In order to calculate the MIRR, three steps are required:

1. Find the *present value* (the cash outflows) of all *negative cash flows* using the assumed borrowing interest rate. (Discount to now.) Call this PV_{out}.

2. Find the *future value* of *all positive cash flows* (the cash inflows) at the end of the project's life, using the assumed investment interest rate. Call this C.

3. Calculate *the rate of return (i)* that will discount the future value C back to the present value PV_{out} for the life time (n) of the project. That is,

$$
i = \left(\frac{C}{PV_{out}}\right)^{\frac{1}{n}} - 1.
$$

This last formula is, of course, simply the formula that we derived in section 6.2 above to deal with a single investment I yielding a single return C at the end of year n.

Let us apply these steps to see how they work.

Example 6.7

An investor has the opportunity to invest in a property development with the following cash flows:

Time (years)	Cash flow (R)
0	$-80\,000$
1	$-10\,000$
2	$10\,000$
3	$35\,000$
4	$135\,000$

Assuming that he can borrow funds at 18% per annum, whereas for an investment with comparable risk he can expect to invest at 15% per annum, what is the MIRR for the development?

(a) Calculate the present value of the negative cash flows.

$$\begin{aligned} PV_{out} &= 80\,000 + 10\,000 \times (1+0{,}18)^{-1} \\ &= 88\,475. \end{aligned}$$

The present value of the negative cash flows is R88 475.

(b) Calculate the future value of the positive cash flows at the end of the project:

$$\begin{aligned} C &= 10\,000(1+0{,}15)^2 + 35\,000(1+0{,}15)^1 + 135\,000 \\ &= 188\,475 \end{aligned}$$

The future value of the positive cash flows is R188 475.

(c) Calculate the MIRR.

$$\begin{aligned} \text{MIRR} &= \left(\frac{C}{PV_{out}}\right)^{\frac{1}{n}} - 1 \\ &= \left(\frac{188\,475}{88\,475}\right)^{\frac{1}{4}} - 1 \\ &= 20{,}81\% \end{aligned}$$

The modified internal rate of return is 20,81%.

Since this is better than the 15% he could get for comparable risk investments, he will presumably invest. On the other hand, if the positive cash flow in the last year dropped to R100 000, the MIRR will be only 14,76% and he might not invest. The point is that this calculation is sensitive to the compound rates. Consequently, one must have good reasons for selecting these.

An investor who is prepared to take a risk expects a rate that is considerably better than the going money market rate. In fact, that is the name of the game – investment implies risk, and the more you hope to make, (generally) the greater the risk.

To end this chapter, a final exercise.

Exercise 6.6

Mr Big Guy is considering a factory development that will involve the following cash flows:

Time (years)	Cash flow (R'000)
0	−760
1	−80
2	300
3	350
4	−180
5	600
6	650

Assuming that he can borrow funds at 16% per annum, and that he expects to receive 19% per annum on investments of comparable risk, what is his MIRR for the project?

6.5 Summary of Chapter 6

The calculation of *internal rates of return* and *net present value* for use in capital budgeting was explained in this chapter. To put their use in perspective, the classical *payback* and *average rate* of *return methods* of capital budgeting were described briefly, and their shortcomings pointed out.

The *internal rate of return* (IRR) was defined as the interest rate which, when applied to sequence of future cash flows, will yield a present value equal to the initial investment outlay. Its use when choosing between alternative investment proposals was described.

The *net present value* (NPV) of an investment proposal was defined as the present value of all future cash flows, less the investment outlay. Its use for selecting investment alternatives was also described.

The related concept of the *profitability index* (PI) was defined as the ratio of the present value of all cash inflows to the present value of all cash outflows.

Finally, the *modified internal rate of return* (MIRR) was defined as the rate that relates the total future value of the positive cash flows (calculated using a compounding rate corresponding to an investment rate for investments of comparable risk) to the present value of the negative cash flows (calculated with a interest rate corresponding to current money market rates).

6.6 Evaluation exercises

1. An investor must select between three alternative proposals: A, B and C. The initial investment outlays and the cash inflows for each are set out in the table below. His cost of capital is a high $K = 21\%$.

   Use, the internal rate of return, the net present value and the profitability index respectively, to advise him on the three proposals. All funds are in R1 000s.

Year	Proposal A Investment: 760	Proposal B Investment: 800	Proposal C Investment: 800
	Cash inflow	Cash inflow	Cash inflow
1	300	310	400
2	330	310	420
3	330	310	440
4	300	310	

2. Projects X and Y have cash flows (in R′000) as indicated below.

Years	0	1	2	3	4	5
X	−800	−40	200	510	510	510
Y	−600	−200	440	440	440	

   Which project appears to be more profitable if the MIRR criterion is used.

   The applicable interest rate to borrow money is 22% per year while an investment can earn 19% per year.

Bonds and debentures

Outcome of chapter

To master the underlying concepts and pricing mechanism of bonds and related financial instruments.

Key concepts

- ✓ Money market instruments
- ✓ Bonds
- ✓ Debentures
- ✓ Face / redemption / nominal value
- ✓ Maturity date
- ✓ Coupon
- ✓ Interest dates
- ✓ Settlement date
- ✓ Yield to maturity

- ✓ R% convention
- ✓ All-in-price
- ✓ Ex interest
- ✓ Cum interest
- ✓ Accrued interest
- ✓ Clean price
- ✓ Discount
- ✓ Premium
- ✓ Par

Study unit 7.1

7.1 Money market instruments

After working through this section you should be able to

▷ give examples of typical money market instruments.

The financial system of a country consists of lenders of funds (government, companies, individuals, etc); borrowers of funds (from the same group as the lenders); and financial institutions (banks, insurance houses, pension funds, etc) that match the funds of the lenders to the requirements of the borrowers.

The financial system facilitates the lending and borrowing of money by the provision of money market instruments.

The types of financial instruments that most of us have to deal with in our capacity as private individuals are generally nonmarketable. These include savings deposits,

personal loans, fixed deposits, mortgage loans and other lease agreements. By "non-marketable" I mean that you cannot trade with them, that is, sell them or buy them from a third party.

Now the idea of selling (or buying) something like a loan sounds crazy but this is essentially what the so-called "money market" is all about. The players in this market are the corporate sector (ie large companies and corporations), the government sector (from central to local), banks and other financial institutions such as insurance houses. And the things in which they deal – the so-called "money market instruments" – apart from the loans and deposits with which you are familiar, constitute a wide spectrum from short-term instruments such as bankers' acceptances (BAs) and treasury bills (TBs), to the very long-term instruments that are typically bonds and debentures. We will discuss the former, and more particularly their pricing (ie the determination of their present value), by using discounting.

7.2 Basic concepts

After working through this section you should be able to
▷ define and describe the basic elements that characterise bonds

One way in which government or semi-government authorities or companies can raise a large sum of money for a good number of years' is to issue *bonds* (as they are called in the case of government loans) or *debentures* (in the case of company loans). What we refer to as the "loan" on our house is, as we saw in the previous chapter, more correctly a "mortgage loan".

Definition 7.1 *Bonds (and debentures) normally pay interest at a fixed rate for a number of years' until the maturity date, and then the principal is repaid. Specifically, a bond (or debenture) is a written contract between the issuer (borrower) and the investor (lender) that states the following:*

1. *The face value (also called the redemption or nominal value) of the bond in rand.*

2. *The maturity date (also termed the redemption date), which is the date when the bond will be repaid (redeemed).*

3. *The coupon rate, which is the rate of interest that the bond pays on its face value, at half-yearly intervals, up to and including the maturity date. (Other intervals are possible but not common, and we shall only consider the half-yearly case.)*

4. *The specific interest dates, twice per year, on which the coupon will be paid, are also stated.*

Study

unit

7.2

⌈ **Example 7.1**

   If an investor purchases R1 000 000 face value of the Bond E168 issued by Eskom
   with a maturity date of 1 June 2012 and a coupon rate of 11% per annum that
   is payable on 1 June and 1 December each year, then, as from the date of
   purchase, he will receive R55 000 (ie $\frac{1}{2}$ of 11% of 1 000 000) on 1 June and 1
   December each year up to, and including, 1 June 2012. In addition, on the
   maturity date he will also receive the redemption value of R1 000 000.

   Note that he will probably not have paid R1 000 000 for the bond – he could
   have paid less or more, depending on the market, as we shall see below.

                                                                            ⌋

Now, since bonds pay interest periodically and can be redeemed on face value at
maturity, the owner of a bond has an asset. Moreover, unlike, say, a lease agreement,
it can be traded – that is, the owner can sell it. This raises the question – at what
price? In other words, what is a particular bond worth?

The answer to this question obviously depends, as it does in the case of any in-
vestment, on the returns the buyer expects to receive in the future by virtue of his
ownership of the bond.

NB
It is apparent that there are three sources of future returns for the owner, namely

1. the redemption value received at maturity

2. the periodic coupon payments until maturity

3. the interest that will be generated by the re-investment of the coupon payments

In order to determine the current price, the present values of these future cash flows
must first be determined and then added together to find the total present value of all
the cash flows. Obviously, the price that should be paid is simply this present value.
However, one question still remains. What interest rate should we use to determine
the present value? We consider this question in the next example, which, hopefully,
will clarify what I have just said.

⌈ **Example 7.2**

   Consider the following hypothetical Bond AAA that John Sinclair wants to buy:

Nominal value	R1 000 000
Coupon rate (half-yearly)	13% per annum
Interest dates	15 March and 15 September
Maturity date	15 March 2014
Settlement date	15 March 2013

   The *settlement date* is the day on which the deal is settled – that is, the day on
   which John must pay for the bond. (There are rules stating what settlement dates are
   allowable. For the sake of simplicity, we shall ignore them for the moment.)

Notice that, in this case, the bond has exactly one year to run to maturity (I have also chosen this to simplify the issue for the moment). The cash flows that John will receive are as follows:

On 15 September 2013: $\frac{1}{2} \times 13\% \times 1$ million $= 65\,000$

John receives R65 000.

On 15 March 2014: $\frac{1}{2} \times 13\% \times 1$ million $= 65\,000$

John receives R65 000.

John will also receive the redemption value of R1 000 000.

Note that John receives no coupon payment on 15 March 2013 (the settlement date). By convention, which we shall discuss a little later, this goes to the seller.

NB

In order to determine the present value of these cash flows, we need to know what return, that is, interest rate, John would like to receive on his money. Suppose this is 15% per annum, compounded half-yearly; in other words, for every R100 invested, he would like to get back R115,5625 in a year. (Check!)

What will John in fact get back on 15 March 2014 as the owner of the bond? This is easily calculated:

In six months, on 15 September 2013, he will receive R65 000, but he can reinvest this for the remaining six months at, in his opinion, 15% per annum. On 15 March 2014, this first coupon payment will be worth:

$$\begin{aligned} 65\,000 \times (1 + \frac{1}{2} \times 15\%) &= 65\,000 \times 1{,}075 \\ &= 69\,875 \end{aligned}$$

The first coupon payment will be R69 875.

The second coupon payment will be R65 000.

On the maturity date 15 March 2014, John will receive the second coupon payment of R65 000 as well as the redemption value of R1 000 000. In total R1 065 000.

The accumulated amount one year after purchase is the first coupon payment of R69 875 plus the second coupon payment and the redemption value of R1 065 000.

The total accumulated amount is R1 134 875 ($69\,875 + 1\,065\,000$).

Now, in order to achieve his desired return of 15% per annum, compounded half-yearly, he must pay the present value of this amount. That is, he must pay

$$1\,134\,875 \times \left(1 + \frac{0{,}15}{2}\right)^{-2} = 982\,044{,}35.$$

John must pay R982 044,35 in order to achieve his desired return of 15%.

What this means, as you have seen, is that whether he purchases the bond for this price, or whether he invests the same sum of money in a bank at 15% per annum, compounded half-yearly, at the end of the year he will, in both cases, have accumulated R1 134 875. We therefore say his *yield* on the bond would be 15% per annum (compounded half-yearly) and, furthermore, that the *yield to maturity* of the bond is 15% per annum (the half-yearly compounding being implied).

Finally, note that, although we first calculated the total sum accumulated in order to determine the price, this was not strictly necessary. What we actually need is the present value of future cash flows, using John's desired yield for discounting.

$$PV(\text{First coupon}) = 65\,000\left(1 + \frac{0{,}15}{2}\right)^{-1}$$

$$PV(\text{Second coupon}) = 65\,000\left(1 + \frac{0{,}15}{2}\right)^{-2}$$

$$PV(\text{Nominal value}) = 1\,000\,000\left(1 + \frac{0{,}15}{2}\right)^{-2}.$$

$$\begin{aligned}
\text{Price} &= 65\,000\left(1 + \frac{0{,}15}{2}\right)^{-1} + 65\,000\left(1 + \frac{0{,}15}{2}\right)^{-2} + 1\,000\,000\left(\frac{1 + 0{,}15}{2}\right)^{-2} \\
&= 982\,044{,}35
\end{aligned}$$

The price is R982 044,35.

The important concept to emerge from the above example, apart from the actual pricing mechanism, is that of the *yield to maturity* of the bond. This is in fact the current anticipated rate of investment return (or yield) that the market foresees for the particular bond. And, since the market price is determined by the buyers and sellers in it, it is in fact the anticipation of the "average investor" (whoever he or she may be!).

In the next section we shall follow up the above results by examining the pricing of a bond in general.

Study
unit
7.3

7.3   The price of a bond on an interest date

After working through this section you should be able to

▷ explain, know and apply the formula (in R%) for pricing a bond on an interest date for a given yield to maturity.

In the previous section we saw that we could simply calculate the price of the bond in the example by using the anticipated yield, the so-called "yield to maturity", to

1. discount all cash flows to their present value,

2. and then add together these present values to obtain the price.

This is the way in which we calculate the price in general.

You might wonder how we obtain the yield to maturity but, for daily trading, such as for bonds listed on BESA (Bond Exchange of South Africa), it is in fact the yield to maturity that is quoted.

Example 7.3

Jill owns the following bond:

Bond BBB

Nominal value	R1 000 000
Coupon rate (half-yearly)	9,25% per annum
Interest dates	30 June and 31 December
Maturity date	31 December 2029
Yield to maturity	15,9% per annum
Settlement date	30 June 2013

She intends to sell it to Jack. Calculate the price on 30 June 2013 that Jack has to pay to Jill for this Bond BBB.

In other words, you are asked to calculate the price on 30 June 2013, given that the yield to maturity is 15,9% per annum.

Let us examine the cash flows.

Remember that the buyer, Jack, does not receive the coupon payment due on 30 June 2013, since, by convention, it belongs to the seller, Jill.

The coupon payments that will be received every six months for 33 half-yearly intervals are:
$$\frac{1}{2} \times 9{,}250\% \times 1\,000\,000 = 46\,250$$

The coupon payments to be received are R46 250.

The coupon payments are represented on the time line.
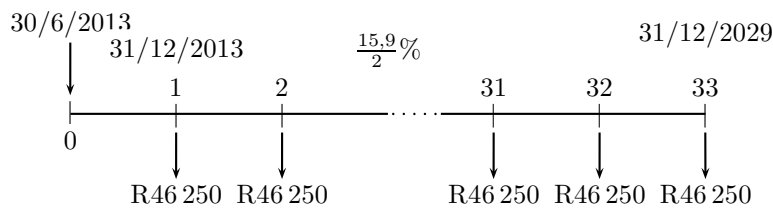
This stream of coupon payments obviously constitutes an ordinary annuity certain. The present value of this stream must be evaluated using the yield to maturity of 15,9% per annum, but with six-monthly discounting intervals.

$$
\begin{aligned}
P_c &= 46\,250 a_{\overline{33}|0{,}159 \div 2} \\
&= 535\,159{,}79.
\end{aligned}
$$

The present value is R535 159,79.

In addition, the buyer, Jack, receives the nominal or face value of R1 000 000 at maturity.

$$
\begin{aligned}
P_f &= 1\,000\,000(1 + 0{,}0795)^{-33} \\
&= 80\,103{,}72
\end{aligned}
$$

The present value of the R1 000 000 is R80 103,72.

The current price is simply the sum of these two present values:

$$P = P_c + P_f = 615\,263{,}51$$

The price is R615 263,51.

NB: Notice that the price consists of two components, that is,

(a) the present value of the stream of coupon payments, and

(b) the present value of the face value.

For both components, the quoted yield to maturity is used to discount the future cash flows in order to determine their present values.

Jill will receive R615 262,51 on 30 June 2013 as well as the coupon of R46 250 that falls due on 30 June 2013. The reason is that the settlement date and coupon date are the same. (See NB, p 106 and Study unit 4, p 110.)

Exercise 7.1

Calculate the price of the following bond:

Nominal value	R1 000 000
Coupon rate (half-yearly)	13% per annum
Interest dates	15 August and 15 February
Maturity date	15 August 2031
Yield to maturity	16,22% per annum
Settlement date	15 August 2013

The R% convention

So far, we have worked only with bonds that have a nominal (or face) value of R1 000 000. This nominal value is in fact quite common. Nevertheless, in principle, any nominal value may occur. To simplify matters, it is customary to work in nominal values of units of R100 and to express the price as the number of rand per R100 unit. Thus, for example 7.3 above, we would write the price as R61,52635% and, for exercise 7.1, the answer would be R81,34645%. (Note that the convention requires five decimal places, not six, which implies rounding to the nearest 10 cents.) Then, **to obtain the actual price for any nominal value, we simply multiply the R% price by the number of R100 units in the nominal value** (10 000 in the two examples).

We can summarise the results of this section in terms of a formula.

NB

For a Bond XXX with

Nominal value	R100
Coupon rate (half-yearly)	c% per annum
Interest dates	Six months apart
Maturity date	dd/mm/yyyy
Yield to maturity	y% per annum
Settlement date	Exactly n half-years before maturity

the price P of the bond or debenture on an interest date in R%, is given by

$$P = \frac{c}{2} \times a_{\overline{n}|y \div 2} + 100 \left(1 + \frac{y}{2}\right)^{-n}$$

or

$$P = d \times a_{\overline{n}|z} + 100(1+z)^{-n}$$

where $d = \frac{c}{2}$ and $z = \frac{y}{2}$.

The coupon equals $c\%$ of the nominal value

$$d = \frac{c}{2}$$

and

$$z = \frac{y}{2}.$$

Thus for Bond ABC with

Nominal value	R100	(for one coupon)
Coupon rate	13% per annum	
Yield to maturity	15,9% per annum	

$$
\begin{aligned}
c &= 13\% \\
&= \tfrac{13}{100} \times \tfrac{100}{1} \quad \text{(Coupons come in R100 units)} \\
&= 13
\end{aligned}
$$

and

$$
\begin{aligned}
d &= \tfrac{13}{2} \\
&= 6{,}5 \\
y &= 15{,}9\% \text{ per year} \\
z &= \tfrac{y}{2} \\
&= \tfrac{15{,}9}{100} \times \tfrac{1}{2} \\
&= 0{,}0795.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
P &= da_{\overline{n}|z} + 100(1+z)^{-n} \\
&= 6{,}5a_{\overline{n}|0{,}0795} + 100(1+0{,}0795)^{-n}.
\end{aligned}
$$

7.4 All-in price

After working through this section you should be able to

▷ know and be able to apply the steps to determine the price of a bond on any settlement date for a given yield to maturity.

In the previous section we derived a formula for the price of a bond or debenture on an interest payment (ie coupon) date (when the term to maturity corresponds to a whole number of half-year intervals). In practice, of course, bonds and debentures may be traded on any date, and must be priced accordingly. (We will qualify the allowable settlement dates later – for the moment, we assume that any date is acceptable.)

In principle, the calculation of the price between interest dates is straightforward.

1. Simply calculate the price at the forthcoming interest date.

2. Then discount this price back to the settlement date.

This is, in fact, the way that the price is calculated, as I shall demonstrate below. There is, however, one complication that must be properly considered, and that is the question of the forthcoming coupon payment. I said above that, when the settlement date coincides with an interest date, as was the case above, then the coupon goes to the seller. But what of, say, a day or a week before?

In order to regulate such situations, so-called "trading rules" apply. These rules, which vary from country to country, state who receives the coupon and when.

NB

In South Africa a bond is sold "cum interest" if the settlement date is ten days or more from the next coupon date. That means that the coupon must be added to the price of the bond. A bond is sold "ex-interest" if the settlement date is strictly less than ten days from the next coupon date.

Obviously whether it is sold "ex" or "cum" affects the price of the bond.

Some examples will illustrate the relevant principles.

⌐ **Example 7.4**

Bond: XYX

Nominal value	R1 000 000
Coupon rate (half-yearly)	13% per annum
Interest dates	15 January and 15 July
Maturity date	15 July 2029
Yield to maturity	15,9% per annum
Settlement date	24 May 2014

Calculate the price.

Represented on a time line, the cash flows are as follows:
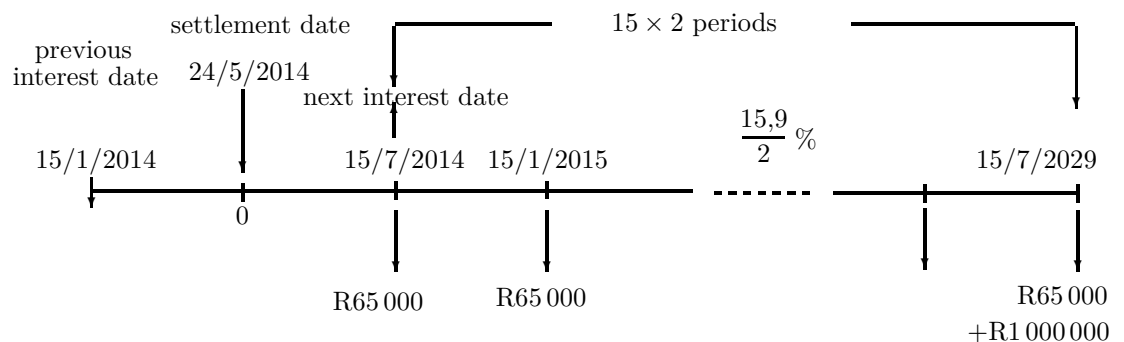
First, we calculate the price on the next interest date using our formula deduced in the previous section:

$$
\begin{aligned}
P &= da_{\overline{n}|z} + 100(1+z)^{-n} \\
P(15/7/2014) &= 6{,}5 \times a_{\overline{30}|0{,}0795} + 100(1+0{,}0795)^{-30} \\
&= 73{,}52215 + 10{,}07675 \\
&= 83{,}59890
\end{aligned}
$$

111

This R83 598,90% is the price on 15 July 2014, which excludes the coupon due on that date. (Notice that we are working in the R% convention.) However, since 24 May is more than ten days from 15 July, the buyer receives the coupon payment of R65 000, or R6,5%. This must be added to the price, which becomes

$$83{,}59890\% + 6{,}5\% = 90{,}09890\%.$$

In other words, this R90 09890% is the price on 15 July 2014, including the value of the coupon due on that date. But the settlement date is 24 May 2014, thus the price on 15 July 2014 must be discounted back to 24 May 2014 to find the current price as at the date of settlement.

At this point, another complication arises. If you think back to section 3.3 on page 34 you will recognise that we are dealing with an odd period calculation here – that is, we have to discount back over a fraction of the half-year interval. The question that arises is: How? In section 3.3 we saw that we could discount using either simple interest discounting or fractional compound discounting – in both cases by counting the exact number of days and expressing it as a fraction of the relevant period. Here, too, we can use either approach. However, the so-called "dealing mathematics conventions" for the Republic of South Africa *prescribe* that we use fractional compound discounting.

We proceed as follows:

First, calculate the remaining number of days from 24 May 2014 to 15 July 2014.

Day number 196 minus day number 144 is 52.

$$R = 52 \text{ days.}$$

Next, calculate the number of days in the half year in which the settlement date falls, namely from 15 January 2014 to 15 July 2014. Day number 196 minus day number 15 is 181.

$$H = 181 \text{ days.}$$

(Note that not all half-years have the same number of days – they vary between 181 and 184 days.)

The fraction of the half year to be discounted back is thus

$$\begin{aligned} f &= \frac{R}{H} \\ &= \frac{52}{181}. \end{aligned}$$

The price on the settlement date of 24 May 2014 is

$$\begin{aligned} P &= P(15/7/2014)(1+z)^{-f} \\ &= 90{,}09890 \times 1{,}0795^{-\left(\frac{52}{181}\right)} \\ &= 88{,}14037. \end{aligned}$$

The price (also known as the *total consideration*) on the settlement date is R88,14037%. It is often referred to as the *all-in price* since it includes the coupon interest (but we shall return to this point below). ⌐

NB

To summarise, the all-in price calculation is quite straightforward:

1. Determine the price on the next interest date.

2. Determine whether the bond is sold cum interest or ex interest; if it is sold cum, add the coupon to the price determined in step 1.

3. Use fractional compounding to discount the price from step 2 back to the settlement date.

You should now be able to do the following two exercises without experiencing any problems.

Exercise 7.2

1. Bond: Iscor Loan 54

Coupon (half-yearly)	14,7% per annum
Interest dates	1 May and 1 November
Maturity date	1 May 2027
Yield to maturity	16,88% per annum
Settlement date	17 April 2015

   Calculate the all-in price.

2. The same bond as for the previous exercise, but with a settlement date of 24 April 2015, that is, seven days later. (Note that this is an ex interest case.)

   NB: No nominal value is given.

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unit
7.5

7.5 Accrued interest and clean price

After working through this section you should be able to

▷ explain the concepts of accrued interest for the ex and cum interest cases;

▷ apply the formula for calculating the accrued interest and the clean price for any settlement date.

Re-examine the answer to the last two exercises. You will recall that in both cases the same bond was quoted at the same yield. The only difference was that the two cases were seven days apart – namely 17 April and 24 April. Yet the difference between the two prices – R95,68106% and R88,65436% respectively – is quite big – R7,02670%. Why the big jump?

The answer to this should be obvious. The first case is "cum interest" whereas the second is "ex interest"; in other words, the first price includes the forthcoming coupon payment while the second does not. And the difference between the two reflects the coupon payment of R7,35% discounted back 14 days.

Cum-interest:

In the case of cum-interest, a coupon has been added to the present value of the bond. As the settlement date falls between two coupon dates, a part of the coupon added does not belong there. We determine this part by using the following formula

$$\frac{H - R}{365} \times c$$

with

  H –   the number of days in the half year in which the settlement date falls –
        thus the number of days between the previous coupon date and
        the coupon date that follows the settlement date.
  R –   the number of days from the settlement date until the coupon date following
        the settlement date.
  c –   the annual coupon rate.

ex-interest:

In an ex-interest case, we use the following formula:

$$\frac{-R}{365} \times c.$$

Clean price:

The clean price of a bond on given settlement date is:
All-in price − accrued interest.

Strictly speaking the accrued interest should be extracted from the all-in price by fractional discounting of the coupon value over the appropriate partial period. However, accepted dealing mathematics for South Africa dictates that the following formulae must be used:

Cum interest:
$$\text{Accrued interest} = \frac{H - R}{365} \times c.$$

Ex interest:
$$\text{Accrued interest} = \frac{-R}{365} \times c$$

NB

An example and a few exercises should help you to grasp the calculations.

⌈ **Example 7.5**

Calculate the all-in price, the accrued interest and the clean price for the following bond for the two given settlement dates.

Eskom Bond E168

Coupon*	11% per annum
Yield to maturity	15,61% per annum
Maturity date	1 June 2031

*__NB__ Unless otherwise stated, it may be assumed that coupons are paid half-yearly on the same day as the maturity date (here 1 June), and six months later (here 1 December).

(a) Settlement date: 31 March 2013

The time line for the cash flows is as follows:
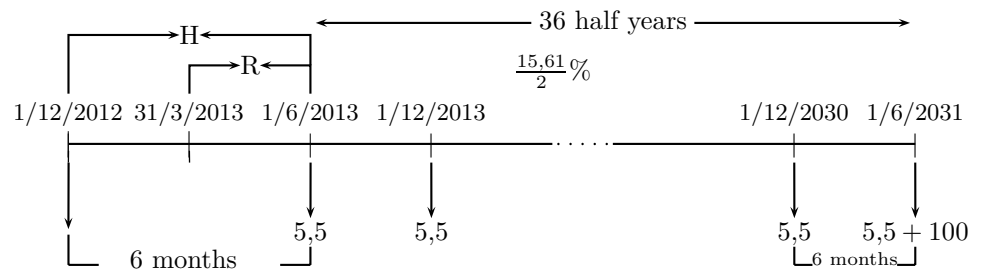
Clearly, this is a cum interest case.

$$P = da_{\overline{n}|z} + 100(1+z)^{-n}$$
$$P(1/6/2013) = 5,5 \times a_{\overline{36}|0,07805} + 100(1+0,07805)^{-36}$$
$$= 72,44142$$

The price at the next interest date is R72,44142%.

Since this is a cum interest case, add the coupon of R5,5% to obtain

$$72,44142 + 5,5 = 77,94142.$$

This value of R77,94142% must be discounted back to 31 March 2013. The remaining number of days from 31/03/2013 to 1/6/2013 is day number 152 minus day number 90 which gives us 62 days.

$$R = 62 \text{ days.}$$

The number of days in the half year from 1/12/2012 to 1/6/2013 is:

Complete the number of days in 2012 and then add the days in 2013. Day number 365 minus 335 plus day number 152 and that gives us 182 days.

$$H = 182 \text{ days.}$$

The fraction of the half year to be discounted back is

$$f = \frac{R}{H}$$
$$= \frac{62}{182}.$$

$$P \;=\; 77{,}94142 \left(1 + \frac{0{,}1561}{2}\right)^{\frac{-62}{182}}$$
$$=\; 75{,}97130.$$

The all-in price is R75,97130%.

Next, we calculate the accrued interest by using the cum interest formula:

$$
\begin{aligned}
\text{Accrued interest} \;&=\; \frac{(H-R)}{365} \times c \\
&=\; \frac{(182-62)}{365} \times 11 \\
&=\; 3{,}61644
\end{aligned}
$$

The accrued interest is R3,61644%.

$$
\begin{aligned}
\text{Clean price} \;&=\; \text{all-in price} - \text{accrued interest} \\
&=\; 75{,}97130 - 3{,}61644 \\
&=\; 72{,}35486
\end{aligned}
$$

The clean price is R72,35486%.

(b) Settlement date: 25 May 2013

This is an ex interest case and the next coupon payment must not be included in the all-in price.

We start from the price at the next interest date, which we calculated above to be
$$P(1/6/2013) = \text{R72,44142\%.}$$

In this case, no coupon is added. The remaining number of days from 25 May 2013 to 1 June 2013 is seven days
$$R = 7.$$

The number of days in the half year is
$$H = 182.$$

The fraction of the half year for discounting is
$$f \;=\; \frac{7}{182}.$$

$$
\begin{aligned}
P(25/5/2013) \;&=\; 72{,}44142 \left(1 + \frac{0{,}1561}{2}\right)^{\frac{-7}{182}} \\
&=\; 72{,}23233
\end{aligned}
$$

The all-in price is R72,23233%.

In this ex interest case, the accrued interest is given by

$$
\begin{aligned}
\text{Accrued interest} \;&=\; \frac{-R}{365} \times c \\
&=\; \frac{-7}{365} \times 11 \\
&=\; -0{,}21096
\end{aligned}
$$

$$
\begin{aligned}
\text{Clean price} \;&=\; \text{All-in price} - \text{Accrued interest} \\
&=\; 72{,}23233 - (-0{,}21096) \\
&=\; 72{,}44329
\end{aligned}
$$

The clean price is R72,44329%.

If you have that uneasy feeling that something is not quite right here, don't despair – you are not alone! I too feel that the formulæ for accrued interest are full of anomalies.

Firstly, as I said above, fractional compound discounting should be applied to extract the accrued interest.

Secondly, if you are going to use simple linear discounting, then it should be done over a half year, not a full year.

Thirdly, the discounting should somehow relate to the anticipated yield since this is the basis for all discounting in the pricing of a bond.

Nevertheless, since, as I intimated above, the question of clean price and accrued interest is mainly of concern to accountants, we accept the prescribed dealing mathematics. From the viewpoint of buyers and sellers of bonds, the important parameter is the all-in price and its basis of calculation is, in my opinion, acceptable.

To end this section, and to test your grasp of the relevant concepts, here is an exercise.

Exercise 7.3

Consider the following Bond ABC:

Nominal (Face) value	R5 000 000
Coupon	5% per annum
Yield to maturity	12% per annum
Maturity date	10 January 2016

Calculate the clean price of the bond on 8 December 2012.

7.6   Discount, premium and par bonds

After working through this section you should be able to

   ▷ explain the differences between discount, premium and par bonds.

1. If   yield to maturity   >   coupon rate
    then   All-in price   <   R100%.

    In this case we say that we purchase the bond at a *discount* (since we buy it for less than its maturity value of R100%).

2. If   Yield to maturity   <   Coupon rate
    then   All-in price   >   R100%.

    In this case we say that we purchase the bond at a *premium* (above its maturity value of R100%).

3. If   Yield to maturity   =   Coupon rate
    then   All-in price   =   R100%.

    In this case, we say that we buy the bond at *par*.

7.7   Summary of Chapter 7

A bond (or debenture) is a written contract between the issuer (borrower) and the investor (lender) that states the following:

1. The *face value* (also *nominal* or *redemption* value)

2. The *maturity date* (also *redemption date*)

3. The *coupon rate* as an annual percentage of the face value

Unless otherwise stated, coupons are paid half-yearly at six-monthly intervals.

The *settlement* date is the day on which a deal, by means of which a bond is bought and sold, is settled. This date should be the third working day after trade (although we have ignored this aspect in our calculations).

The *yield to maturity* of a bond is the current, anticipated rate of investment return (or yield) expressed as a % per annum but compounded half-yearly as determined by the market.

The *price* (P) of a bond in R% of coupon c% per annum and yield y%, on an interest date exactly n half-years before maturity, is given by

$$P = \frac{c}{2} \times a_{\overline{n}|y \div 2} + \frac{100}{\left(1 + \frac{y}{2}\right)^n}$$

or

$$P = d \times a_{\overline{n}|z} + 100(1+z)^{-n}.$$

The concepts "*cum interest*" and "*ex interest*" were introduced.

In general, the all-in price of a bond on any settlement date between interest dates is determined by means of the following steps:

1. Determine the price on the forthcoming interest date (interest date that follows on the settlement date) by means of the above pricing formula.

2. Determine whether the bond is sold cum interest or ex interest; if it is sold cum, add the coupon payment to the price determined in 1.

3. Determine the number of days (R) from the settlement date to the forthcoming interest date, and the number of days (H) in the half year in which the settlement date falls. Use fractional compound discounting to discount the price in step 2 back for the fraction $f = \frac{R}{H}$ of the half year to the settlement date.

"*Accrued interest*" was defined as follows:

$$\text{Cum interest case:} \quad \frac{H-R}{365} \times c$$

$$\text{Ex interest case:} \quad \frac{-R}{365} \times c$$

where c is the full, annual coupon payment.

The "*clean price*" is given by

$$\text{Clean price} = \text{All-in price} - \text{Accrued interest}.$$

Finally, *discount, premium* and *par* bonds were defined.

A bond is sold at *discount* if its price is less than R100%, and at a *premium* if it is greater than that value. If the bond is sold at exactly R100%, then it is sold at *par*.

The cash flows associated with investment in each of these types of bond were analysed and discussed, and it was pointed out that the "*interest rate risk*" of the three differs.

7.8   Evaluation exercises

1. Consider the following Bond E11

Nominal value	R2 500 000
Coupon rate (half-yearly)	13,2% per annum
Redemption date	1 November 2029
Yield to maturity	15,70% per annum

   For each of the settlement dates, that is,

(a) 25 September 2012

(b) 25 October 2012,

calculate the all-in price, the accrued interest and the clean price.

The handling of data

Outcome of chapter

> To master the basic concepts and applications of limited statistics and basic linear regression.

Key concepts

- ✓ Subscripts and summations
- ✓ Arithmetic mean
- ✓ Weighted mean
- ✓ Standard deviation
- ✓ Variance
- ✓ Linear functions
- ✓ Slope

- ✓ Intercept
- ✓ Correlation
- ✓ Regression
- ✓ Scatter diagram
- ✓ Correlation coefficient
- ✓ Coefficient of determination

8.1 Subscript and summation

Study unit 8.1

> *After working through this section you should be able to*
> ▷ understand and use single as well as double subscripts for variables;
> ▷ understand and use single as well as double summation notation.

With today's technology, companies are able to collect tremendous amounts of data with relative ease. Indeed, many companies now have more data than they know what to do with. Retailers collect point-of-sale data on products and customers every time a transaction occurs; credit agencies have all sorts of data on people who have or would like to obtain credit; investment companies have a unlimited supply of data on the historical patterns of stocks, bonds and other securities; and government agencies have data on economic trends, the environment, social welfare, consumer product safety and virtually everything else we can imagine. However, all the collected data are usually meaningless until they are analysed for trends, patterns, relationships and other useful *information*. In many business contexts, data analysis is only the first step in the solution of a problem. Acting on the analysis and the information it provides to make justifiable decisions is a critical next step.

In a business it is very important to solve problems quickly and effectively. However, you cannot solve a problem if you don't know that it exists! A good manager will detect a problem and solve it before it becomes a crisis. But how does he manage to do that? Managers use data to understand their organisation and identify problems. As said, the data collected must be organised and summarised to make it usable.

In this chapter we discuss a few simple but effective, methods to make sense out of available information.

8.1.1   Simple summations

Suppose we have data on five persons' income, savings and assets:

	Person 1	Person 2	Person 3
Income	95 000	74 500	46 700
Savings	2 350	10 600	5 930
Assets	120 200	85 240	240 600

Referring again to the example, if x refers to income, y to savings and z to assets then x_1, x_2 and x_3 refer to the income of the five persons. Similarly, we could refer to the five persons' savings as y_1, y_2 and y_3, and to their assets as z_1, z_2 and z_3. If we want to discuss any one of these numbers in general we shall refer to it as x_i, y_i or z_i where i is, so to speak, a variable subscript which, in this particular example, can take on the values 1, 2 or 3.

Instead of writing the subscript as i we could just as well have used another letter such as j, k, $l \ldots$, and instead of x, y and z we could just as well have used other arbitrary letters or symbols. In general it is customary to use different letters for different kinds of measurements together with subscripts for different individuals (different items). We might thus write $x_2 = $ R74 500, $y_2 = $ R10 600 and $z_2 = $ R85 240 for the second person's income, savings and assets.

Definition 8.1 *In order to simplify formulæ that will involve large sets of numerical data, let us now introduce the symbol Σ (capital Greek sigma, standing for S in our alphabet) which is merely a mathematical shorthand notation for summation. By definition, we shall write*

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$$

which reads: "the summation of x_i, i going from 1 to n." In other words, $\sum_{i=1}^{n} x_i$ stands for the sum of the x's having the subscripts 1, 2, $\ldots$, and n.

Please note that

$$\sum_{i=1}^{n} x_i$$

is exactly the same as $\sum_{i=1}^{n} x_i$ (it's just another way of writing the summation).

According to this definition we might write, for example, the sum of the income as:

$$\sum_{i=1}^{3} x_i \quad = \quad x_1 + x_2 + x_3$$
$$= \quad 95\,000 + 74\,500 + 46\,700$$
$$= \quad 216\,200.$$

Also

$$\sum_{i=1}^{3} y_i^2 \quad = \quad y_1^2 + y_2^2 + y_3^2$$
$$= \quad 2\,350^2 + 10\,600^2 + 5\,930^2$$
$$= \quad 153\,047\,400$$

or

$$\sum_{i=2}^{3} z_i \quad = \quad z_2 + z_3$$
$$= \quad 85\,240 + 240\,600$$
$$= \quad 325\,840$$

beginning in the last example with the subscript 2 and ending with the subscript 3.

8.1.2   Double subscripts and summations

Sometimes we want to add various sums. For example, assume we have the income of five individuals from town A, five individuals from town B and five individuals from town C. We want the total income of all 15 individuals. We may use the letter x for the variable "income" which has two characteristics namely town and individual. If the town is represented with the subscript i ($i = 1, 2, 3$) and the individual with j ($j = 1, 2, 3, 4, 5$) then x_{ij} will refer to the income of a specific individual in a specific town. For example, x_{13} will refer to individual three living in town one (A).

The sum of income of the five individuals of town A is

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = \sum_{j=1}^{5} x_{1j}.$$

In this case, we sum over the five individuals in a specific town. The sum of income of the five individuals of town B is

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = \sum_{j=1}^{5} x_{2j}.$$

The sum of income of the five individuals of town C is

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = \sum_{j=1}^{5} x_{3j}.$$

The total sum of all (15) income is

$$\sum_{j=1}^{5} x_{1j} + \sum_{j=1}^{5} x_{2j} + \sum_{j=1}^{5} x_{3j} = \sum_{i=1}^{3} \sum_{j=1}^{5} x_{ij} = \sum_{j=1}^{5} \sum_{i=1}^{3} x_{ij}.$$

8.2   Data

<div style="border:1px solid;">

After working through this section you should be able to

▷ explain what a population and a sample is;

▷ calculate the arithmetic mean of a set of data;

▷ calculate the weighted mean of a set of data;

▷ calculate the variance and standard deviation of a set of data.

</div>

Study
unit
8.2

In working with data, the word *sample* is used very much in its everyday connotation. Taking of samples is common in everyday practice. For example: the public opinion about one or other important subject is commonly spread by the media after a quick public-opinion research. Samples like this are not always representative of the big totality from which it is taken.

8.2.1   Population and sample

Definition 8.2 *The population is defined as all the items or things under consideration.*

Definition 8.3 *A sample is a representative group or subset of the population. It is the portion of the population that is selected for analysis.*

Rather than take a complete *census* of the whole population, *sampling procedures* focus on collecting a small representative group of the large population. The resulting sample provides information that can be used to estimate characteristics of the entire population.

If we interview ten of a possible 500 employees of a certain company, we can consider the opinions which they express to be a sample of the opinions of *all* employees; if we measure the "lifetimes" of five light bulbs of a certain brand, we can consider these measurements to be a sample of the measurements we would have obtained if we had measured *all* similar light bulbs made by the same firm. The last illustration indicates why we often have to be satisfied with samples, being unable to get information about the complete population. (If we measured the "lifetimes" of *all* the light bulbs made by the firm, they would be in the unfortunate position of having none left to sell.) Another reason for taking samples is the cost aspect. Sometimes it is too expensive to investigate a whole population and a sample is then the only alternative.

After taking the sample, we want to get all usable information from the data that we have collected. We are going to calculate different measures that will each tell us something different about our data. In Section 8.2.2 and 8.2.3 we are going to discuss

two types of measures of "central values", also called *measures of location.* These measures tell us the location of the "centre" or the "middle" of a set of data, in other words, some sort of average. In Section 8.2.4 we will look at a measure that will tell us something about the extent to which our data are spread, a so-called "measure of variation".

8.2.2 The arithmetic mean

There are many problems in which we would like to represent a set of numbers by means of a *single* number which is, so to speak, descriptive of the entire set of data. The most popular measure used for this purpose is called an *average* or the *arithmetic mean.*

Definition 8.4 *Given a set of n numbers, $x_1, x_2, x_3, \ldots, x_n$, their arithmetic mean is defined as their sum divided by n.*

⌈**Example 8.1**

Let us consider a portfolio of five stocks with the following prices (in rand) per share: 1,46; 1,03; 2,85; 1,56; 3,42. The average price per share is

$$\frac{1{,}46 + 1{,}03 + 2{,}85 + 1{,}56 + 3{,}42}{5} = 2{,}06.$$

The average price is R2,06.   ⌋

NB

Symbolically, we can therefore write

$$\text{arithmetic mean} = \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

where
$\overline{x}$ (read as *x*-bar) is the generally accepted symbol for the arithmetic mean;
n is the number of observations;
Σ is the Greek letter for S and means "sum";
x_i represents the *i*-th observation.

Exercise 8.1

The manager of a bank has recorded 30 observations on the *number of days between re-orders* of customer's cheque books.

The re-order intervals (in days) are:

28	36	25	27	17	37
34	27	20	27	33	39
38	28	20	33	26	19
22	36	15	22	33	32
34	24	26	36	29	32

Find the arithmetic mean of the *number of days between re-orders.*

Some of the noteworthy properties of the arithmetic mean are that (1) most people understand what is meant by *mean*, although they may not actually call it by that name, (2) it always exists, it can always be calculated for any kind of numerical data; and (3) it is always unique, or, in other words, a set of numerical data has *one and only one* arithmetic mean.

8.2.3   The weighted mean

There are many problems in which we cannot average quantities without paying some attention to their *relative importance* in the overall situation we are trying to describe. For example, if there are two food stores in a given town selling butter at R27,90 a kilogram and R28,10 a kilogram, respectively, we cannot determine the over-all price paid for butter in this town unless we knew the number of kilograms sold by each store. If most people buy their butter in the first store, the average price they pay per kilogram will be closer to R27,90, and if most of them buy their butter in the second store, the average price will be closer to R28,10. Hence, we cannot calculate a meaningful average unless we know the relative weight (the relative importance) carried by the numbers we want to average. Similarly, we might get a wrong picture if we calculate the *mean* of change in the values of stocks without paying attention to the number of shares sold of each stock, and we will not be able to average the number of accident fatalities reported by three different airlines per 1 000 000 passenger miles unless we know the number of passenger miles flown by each airline.

Example 8.2

Let us suppose that in a new town development an estate agent sells three types of small plots for R256 900, R308 900 and R390 900 respectively and that we would like to know the average price he receives per plot. Clearly, we cannot say that he receives an average of R318 900 ((256 900 + 308 900 + 390 900) ÷ 3) per plot, unless by chance he happens to sell an equal number of plots of each type. If we are given the additional information that during a certain month he has sold 60 plots of the cheapest type, 30 plots of the medium-priced type and

10 plots of the most expensive type, we find that he received $60 \times 256\,900 + 30 \times 308\,900 + 10 \times 390\,090 = 28\,590\,000$ for $60 + 30 + 10 = 100$ plots and that he, thus, received, on average, R285 900 per plot. The average we have calculated here is called a weighted mean. We have averaged the three prices, giving a suitable weight to the relative importance of each, namely to the number of plots sold of each type.   ⌐

Definition 8.5 *In general, if we want to average a set of numbers $x_1, x_2, x_3, \ldots, x_n$ whose relative importance is expressed numerically by means of some numbers $w_1, w_2, w_3, \ldots, w_n$ called their weights, we shall use the weighted mean, which is defined by the formula*

$$\overline{x}_w = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}.$$

Looking at our "new town development example" the weighted mean is calculated as follows:

$$
\begin{aligned}
\overline{x}_w &= \frac{\sum_{i=1}^{3} x_i w_i}{\sum_{i=1}^{3} w_i}. \\
&= \frac{x_1 \times w_1 + x_2 \times w_2 + x_3 \times w_3}{w_1 + w_2 + w_3} \\
&= \frac{60 \times 256\,900 + 30 \times 308\,900 + 10 \times 390\,900}{60 + 30 + 10} \\
&= 285\,900.
\end{aligned}
$$

The weighted mean is R285 900.

The selection of weights is not always quite so obvious. For example, if we wanted to construct a *cost-of-living* index, we would have to worry about the roles played by different commodities in the average person's budget. The weights that are most frequently used in the averaging of *prices* are the corresponding *quantities* consumed, sold or produced.

Exercise 8.2

For one year Mr Jones invests R1 000, R800 and R3 200 respectively at 12%, 13% and 15% interest rate per year. Calculate the weighted average interest rate on his investments.

8.2.4   Standard deviation and variance

The mean discussed in the previous two sections provide us with a single number which represents the middle value of a set of data. Although the information contained in a measure of location may well provide an adequate description of a set of data in a limited number of problems, in general we will find it necessary to supplement it by describing additional features. *Measures of variation*, *spread* or *dispersion* tell us something about the extent to which our data are dispersed - the extent to which they are spread out or bunched.

Let us suppose, for instance, that we want to compare two stocks, Stock X and Stock Y, and that over the last six years the closing price of Stock X for each year was such as to yield to maturity

$$6{,}0; \ 5{,}7; \ 5{,}6; \ 5{,}9; \ 6{,}1 \text{ and } 5{,}5\%,$$

while, for the same period of time, the corresponding figures for Stock Y were

$$7{,}2; \ 7{,}7; \ 4{,}9; \ 3{,}1; \ 3{,}4 \text{ and } 8{,}5\%.$$

As can easily be checked, the means of these two sets of percentages are both 5,8 and, hence, if we based our evaluation of the stocks entirely on their average yields, we would be led to believe that, at least for the given six years, they were equally good.

A more careful analysis reveals, however, that, whereas the closing price of Stock X was such that the yield to maturity was consistently close to 5,8% (it varied from 5,5 to 6,1%), the performance of Stock Y was much more erratic. In the fourth year its yield to maturity was as low as 3,1%, and even though this was offset by 8,5% in another year, it is clear that Stock Y provided a much *less consistent* investment than Stock X. In this example therefore we have, a problem in which an intelligent evaluation requires some measure of the fluctuations or variability of our data in addition to a measure of central location such as the mean.

Since the dispersion of a set of numbers is *small* if the numbers are bunched very closely around the mean, and it is *large* if the numbers are spread over considerable distances away from the mean, we might define variation *in terms of the distances (deviations) by which the various numbers depart from the mean.*
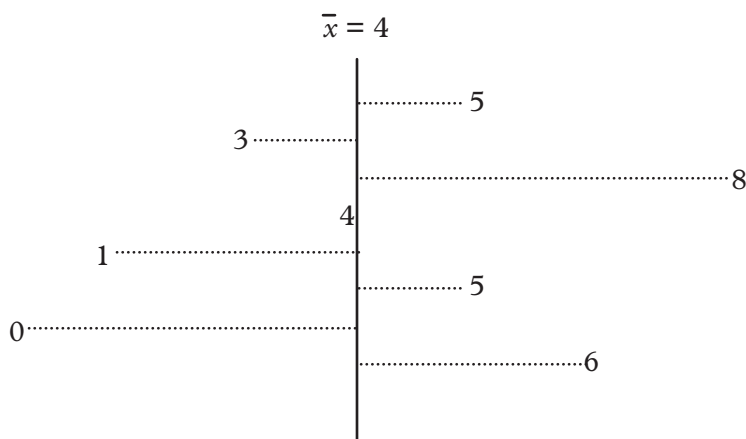
Consider the following data set:

$$5; \ 3; \ 8; \ 4; \ 1; \ 5; \ 0; \ 6.$$

The mean is

$$\overline{x} = \frac{32}{8} = 4.$$

Let's plot the data point around the mean.

This gives a clear picture of how each data point departs from the mean (the spread of the data around the mean). Now we need some quantity to measure this deviation from the mean, and we will call it the *standard deviation*.

Calculate the deviation from the mean for each observation, that is $x - \overline{x}$:

$5 - 4 = 1;\ 3 - 4 = -1;\ 8 - 4 = 4;\ 4 - 4 = 0;\ 1 - 4 = -3;\ 5 - 4 = 1;\ 0 - 4 = -4;$

$6 - 4 = 2$

When you add this you get 0 – which tells you nothing! However, the number crunchers of olden days did not become discouraged, and came up with a clever idea.

Use $(x - \overline{x})$, but square it. The square of any value is always a positive number. The mean of the squared deviations is called the variance.

NB

The positive square root of the variance is called the standard deviation, and we will use this measurement to give an indication of the spread of the data around the mean.

Definition 8.6 *The standard deviation of a sample is defined as:*

$$S = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n - 1}}.$$

Notice that we divide by $n - 1$ and not by n. The reason for this is not covered in this course.

Now back to our sample (remember $\overline{x} = 4$).

x	$(x - \overline{x})$	$(x - \overline{x})^2$
5	1	1
3	−1	1
8	4	16
4	0	0
1	−3	9
5	1	1
0	−4	16
6	2	4
Total		48

Thus

$$\sum_{i=1}^{8}(x_i - \overline{x})^2 = 48$$

The variance is

$$
\begin{aligned}
S^2 &= \frac{\sum_{i=1}^{8}(x_i - \overline{x})^2}{8 - 1} \\
&= \frac{48}{7} \\
&= 6{,}86.
\end{aligned}
$$

The standard deviation is $\sqrt{6{,}86} = 2{,}62$.

PLEASE NOTE: This calculation can be done directly on your calculator. See Tutorial Letter 101 for the key operations for the recommended calculators.

⌐**Example 8.3**

Calculate the standard deviation of the following sample.

Selling price of a specific share over the 5 working days (in cents):
1 630; 1 550; 1 430; 1 440; 1 390.

First calculate the mean, $\overline{x}$.
Then calculate the deviation from $\overline{x}$ for each observation and square it.
Divide the sum of the squares by $n - 1 = 4$.
The standard deviation is the square root of the variance.
We get:

$$\overline{x} = \frac{\sum_{i=1}^{5} x_i}{n} = \frac{7\,440}{5} = 1\,488$$

The sum of the squares of the deviations of each data point from the mean is:

$$
\begin{aligned}
\sum_{i=1}^{5}(x_i - \overline{x})^2 &= 142^2 + (62)^2 + (-58)^2 + (-48)^2 + (-98)^2 \\
&= 20\,164 + 3\,844 + 3\,364 + 2\,304 + 9\,604 \\
&= 39\,280
\end{aligned}
$$

The variance is:

$$S^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1} = \frac{39\,280}{4} = 9\,820$$

The standard deviation is:

$$S = \sqrt{9\,820} = 99{,}10$$

⌐

The calculations can be done directly on your calculator.

Study unit 8.3

8.3 Describing relationships

Note

This study unit is purely for background knowledge for Study unit 4.

After working through this section you should be able to

▷ define a linear function;

▷ determine the intercepts on the axes of a straight line;

▷ determine the slope of a straight line;

▷ determine the equation of a straight line from two points – the points can be given or you should be able to unravel them from information;

▷ draw a graph of a straight line.

The foremost objective of many investigations of data in business and economics is to predict - that is, to forecast such things as the potential market for a new product, the future value of a property, the growth of an industry, or over-all economic conditions. Forecasting is a process where relationships between variables are described, that is describing relationships between quantities that are known and quantities that are to be predicted, in terms of mathematical equations.

One of the simplest and most widely used equations for expressing relationships in various fields is the linear equation or also called a *straight line*.

8.3.1 Linear functions

Linear equations are useful and important not only because there exist many relationships that are actually of this form, but also because they often provide close approximation to complicated relationships which would otherwise be difficult to describe.

NB

The general expression of a linear equation or a straight line is

$$y = a + bx$$

where a and b are numbers.

Once a and b are known we can calculate a predicted value y for any given value of x by direct substitution. Note that the value of y is dependent on the value of x. If

the value of x changes, the value of y will also change. That is why y is called the dependent variable and x the independent variable. Thus, a linear function assigns one value of y to each value of x. In this way a set of ordered pairs of data which we can write as $(x; y)$ is established. Each of these pairs corresponds to a point on a graph, and if we plotted all the points we would obtain what is called a graph of a given function.

8.3.2　The set of axes

To represent a linear function graphically, we use a set of axes.

Draw two lines at right angles to each other as shown. The **horizontal line** is generally used for the **x**-values or the **independent variable** and the **vertical line** for the **y**-values or **dependent variable**. The point where they cross is the common **origin**. A convenient **scale**, which need not be the same for both lines, is indicated on each. In the same way as it is customary to associate points to the right of the origin with positive values of the independent variable, and points to the left with negative values, so it is customary to associate points above the origin with positive values and points below it with negative values of the dependent variable. Furthermore, it is customary to refer to the horizontal line as the **x-axis** and the vertical line as the **y-axis**. A scaled set of axes introduced in this way is referred to as a **rectangular coordinate system**. As indicated in the figure, it divides the plane (that is the sheet of paper) into four sections which are known as **quadrants** and which are numbered as shown.
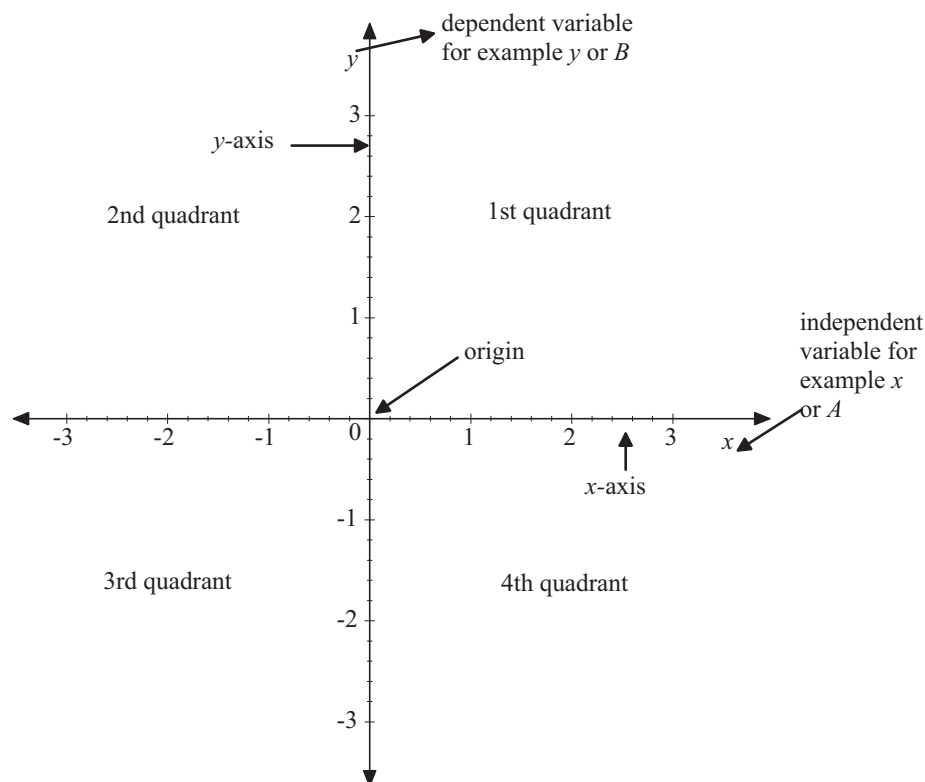
If x is the independent variable and y the dependent variable, then x and y are

both positive in the first quadrant. In the second quadrant x is negative and y is positive, in the third quadrant x and y are both negative, and in the fourth quadrant x is positive and y is negative. Since most business problems deal with positive quantities, we shall be concerned mainly with points in the first quadrant, but if we regard losses as negative profits, deductions as negative additions, deficits as negative income, etcetera, we shall also have the occasion to work with points in the other three quadrants. Whenever you "read" a graph, you carefully establish the variables represented on each axis and the relevant scales. **Always label the axes clearly**. Remember the variables you want to draw on the axes need not be x and y, but any variables, for example A and B or x_1 and x_2.

8.3.3   The intercepts of a straight line

The general equation for a straight line is:

$$y = a + bx$$

The point where the ***line cuts the y-axis***, is called the ***y-intercept***. This is where $x = 0$. At this point the value of y is

$$y = a + b \times 0 = a.$$

The point where the ***line cuts the x-axis***, is called the ***x-intercept*** or ***root***. The value of the ***intercept on the x-axis*** is where $y = 0$, this is where $a + bx = 0$.

$$x = -\frac{a}{b}$$

8.3.4   The slope of a straight line

The steepness with which straight lines ascend or descend is called the slope.

The slope, b is the ratio of the change in y-values to a given change in x-values. It is also called the slope coefficient.

NB

In terms of two arbitrary points, P_1 with coordinates $(x_1; y_1)$ and P_2 with coordinates $(x_2; y_2)$ on the straight line, we can write

$$\text{slope} = b \quad = \quad \frac{\text{change in } y\text{-value}}{\text{change in corresponding } x\text{-value}}$$
$$= \quad \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of a straight line is depicted in the following graph:
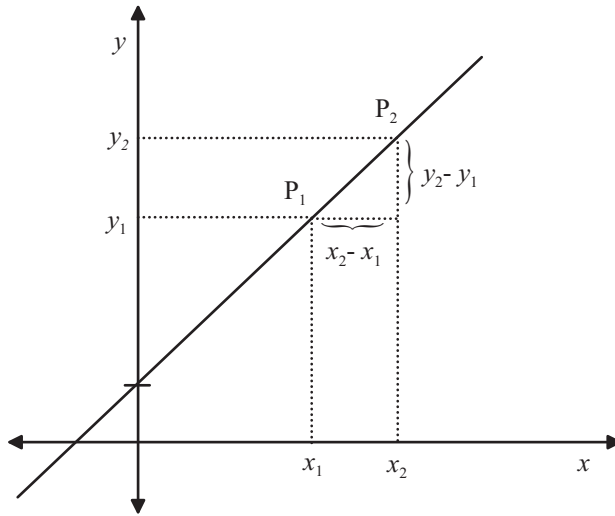
Now it is clear that b is a measure of the steepness of a straight line. The greater the change in y for a given change in x, the steeper the line.

8.3.5   Using two points to determine the equation of a straight line

We use the following formula to determine the equation for a straight line passing through the points $(x_1; y_1)$ and $(x_2; y_2)$:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 8.4

Determine the expression for the straight line passing through the points $(1; 3)$ and $(3; 7)$.

Substitute these values into the formula:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 3}{x - 1} = \frac{7 - 3}{3 - 1}$$

$$\frac{y - 3}{x - 1} = \frac{4}{2}$$

$$
\begin{aligned}
y - 3 &= 2\,(x - 1) \\
y &= 2x - 2 + 3 \\
&= 2x + 1
\end{aligned}
$$

Sometimes the two points are not given to you, but you must unravel them from the information given, as illustrated below.

Example 8.5

In a company, the average monthly sales of company salespeople is linearly related to the number of hours a person has spent in formal sales-training courses. If a person has spent 60 hours in training, his average monthly sales will be R80 000. If a person has spent 20 hours in training, his average monthly sales will be R60 000. Determine the linear sales function.

The question leads us to conclude that the monthly sales of company salespeople is dependent on the number of hours they have spent in formal sales-training courses.

Let $x =$ the number of hours in training which is also the independent variable. Let $y =$ monthly sales or the dependent variable.

The following data for x and y are given:

x	y
60	80 000
20	60 000

Thus two data points that satisfy the linear relationship are $(60; 80\,000)$ and $(20; 60\,000)$.

By substituting the above data into the following formula we get:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 80\,000}{x - 60} = \frac{60\,000 - 80\,000}{20 - 60}$$

$$\frac{y - 80\,000}{x - 60} = \frac{-20\,000}{-40}$$

$$
\begin{aligned}
y - 80\,000 &= 500\,(x - 60) \\
y &= 500x - 30\,000 + 80\,000 \\
&= 500x + 50\,000
\end{aligned}
$$

The equation for the line passing through the two points $(60; 80\,000)$ and $(20; 60\,000)$ is therefore

$$y = 50\,000 + 500x.$$

You can also use your calculator to determine the equation for the straight line.

8.3.6   Representing a function on a set of axes

Only two points are needed to determine an equation of a straight line. If you are not convinced of this, just mark two points on a piece of paper and try to put more than one straight line through them. To draw the graph of a straight line we make use of the general method:

1. Draw your axes and label them.

2. Choose the scale of the axes.

3. Plot the two given points.

4. Draw a line through the two plotted points.

⌐Example 8.6

Graph the straight line that passes through the points $(1; 4)$ and $(4; 2)$.

The graph follows.
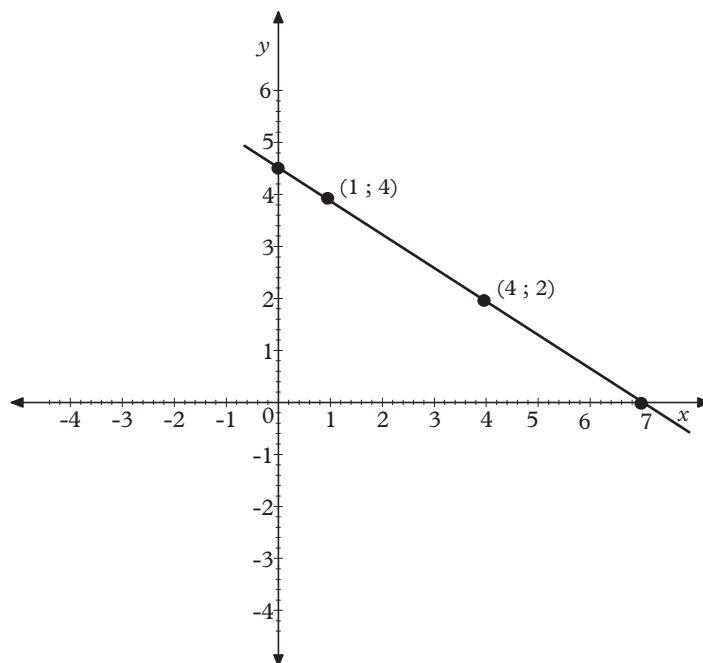
8.4 Correlation and regression analysis

After working through this section you should be able to

▷ define the concepts correlation and regression;

▷ draw a scatter diagram;

▷ calculate the Pearson's correlation coefficient and coefficient of determination;

▷ determine the equation of the regression line for given data;

▷ use the correlation coefficient to determine how well the estimated line fits the data.

The purpose of correlation and regression analysis is to measure the relationship between two variables. Such a relationship, if it can be determined, can be used effectively when making estimates and forecasts. It is, however, important to distinguish between correlation and regression. Regression has to do with quantifying the underlying structural relationship between the variables; in other words, it defines the exact form of the relationship (straight line, curves, etc), while correlation investigates the strength of the identified relationship between variables and quantifies it (as excellent, good, reasonable or weak).

Correlation and regression are widely applied in the business world. Such applications could, for example, be used to forecast the future behaviour of the market or to establish how different factors could influence the market.

A study of these analytical methods begins by looking at the use of graphic representation and the numerical measurement of correlation. We then examine the method used to describe the regression relationship between two variables.

8.4.1 Correlation analysis

8.4.1.1 Scatter diagrams

As a first step for ascertaining whether there is a relationship between two variables, we can draw a graph of the available data. Scatter diagrams give a visual indication of the relationship between two variables and indicate in what direction further analyses should be done. A scatter diagram consists of a number of points that are plotted on a graph so that every point represents a data pair. A scatter diagram is drawn by plotting $(x; y)$ observations on a graph where the *independent variable*, x, is indicated on the horizontal axis and the *dependent* variable, y, on the vertical axis.

A visual investigation of the possible relationship between the two variables x and y provides us with a first impression of the probable regression and correlation results.

⌐**Example 8.7**

Suppose a sales manager has records containing data on the annual sales volumes and the number of years' experience of his sales staff. The information is summarised in the table below.

Sales person	Years of experience	Annual sales (in R1 000s)
1	1	90
2	2	107
3	3	102
4	4	112
5	6	113
6	8	121
7	10	129
8	10	133
9	11	134
10	13	136

Plot the data on a graph with the number of years' experience on the horizontal axis and the annual sales on the vertical axis. This is called a scatter diagram because it consists of points that are scattered over the graph or diagram.
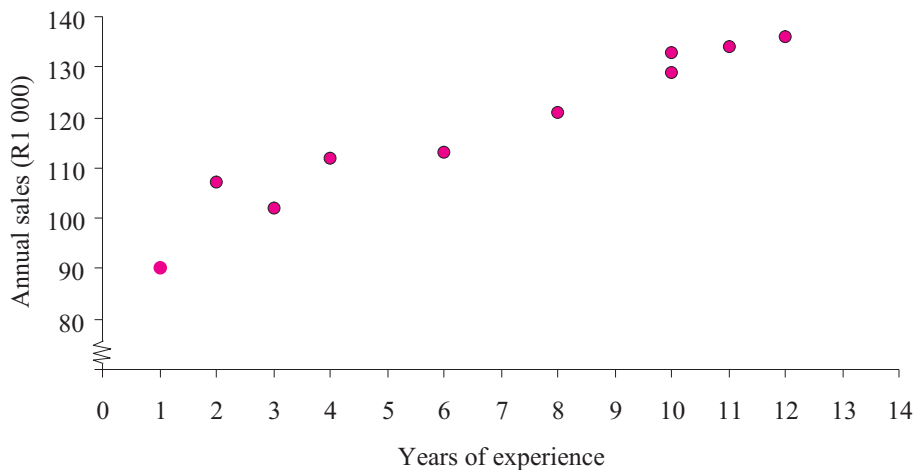
It is clear that a relationship exists between x and y, because y, in general, will increase if x increases. There are, however, instances where x increases but not y. In order to interpret such diagrams, they can be compared to extreme cases, such as if there were a perfect relationship or no relationship at all. Examples are given below.
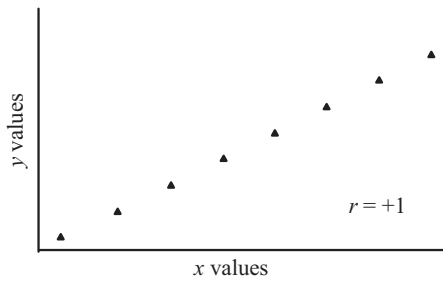
(a) $r = +1$: Perfect positive linear correlation
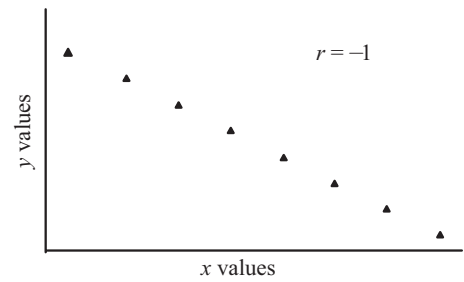
(b) $r = 1$: Perfect negative linear correlation
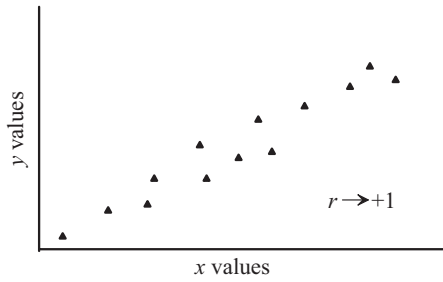
(c) $0 < r < +1$: Positive linear correlation
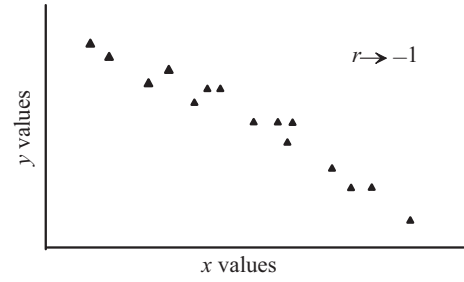
(d) $1 < r < 0$: Negative linear correlation
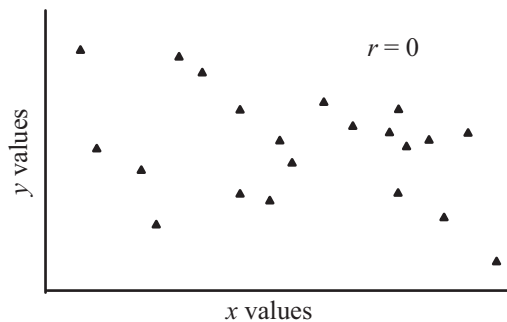
(e) $r = 0$: No linear correlation

Graph (a) shows a perfect positive linear relationship, in other words, it is an example of how increases in the value of the independent variable, on the x axis, cause the values of the dependent variable on the y axis to increase as well. Graph (b) shows a perfect negative linear relationship that occurs when increases in the value of the independent variable lead to decreases in the value of the dependent variable. Graph (c) shows a positive linear relationship and graph (d) a negative linear relationship. Graph (e) shows a case where there is absolutely no relationship between the two variables.

8.4.1.2   Correlation coefficient

There are situations where the decision maker is not as concerned about the comparison that relates the two variables to each other as the degree to which the two variables are related. In such cases, a statistical technique known as *correlation analysis* can be used to determine the strength of the relationship between the two variables.

The result of a correlation analysis is a number, which is called a correlation coefficient. The correlation coefficient will always have a value between -1 and $+1$. A

value of $+1$ means a perfect positive correlation and again corresponds with the situation where all the points on the scatter diagram lie on a straight line with a positive slope. A value of -1 means perfect negative correlation and once again corresponds to the situation where all the points on the scatter diagram lie on a straight line with a negative slope. If the correlation coefficient is close to $+1$ or -1, the correlation is high, and if it is close to 0, the correlation is low. If the coefficient of linear correlation is zero, we say that there is no linear correlation, or that the two variables involved are not linearly related.

Note

If the correlation coefficient indicates that there is no *linear* relationship, it still does not mean that there is no relationship between the variables.

Let us look once again at the graph. In case (a), $r = +1$ and in case (b), $r = -1$. In case (c), r will lie between 0 and $+1$, and in case (d) between -1 and 0. In case (e), $r = 0$.

In other words, a variable that has a great influence on the sales figures of an enterprise will show a high correlation with sales. A high correlation coefficient is not, however, proof that a relationship between the variables exists. The correlation coefficient is just a mathematical measurement of relationships and does not imply a cause and effect situation (in which changes in the one variable result in changes in the other) between variables. Even if such a causal relationship is suspected, it is not to say that the enterprise necessarily would be able to control the causal variable or future sales directly (or solely) through it.

Two correlation coefficients are generally used: Spearman's rank correlation coefficient (used for ordinal data) and Pearson's correlation coefficient (used for quantitative data). We will only deal with the Pearson's correlation coefficient.

■ Pearson's correlation coefficient

When quantitative data are involved, Pearson's correlation coefficient is used. Since the data are measurable, it is possible to say by how much one variable is better than the other, and a more reliable result can be obtained. Such data can be used to do further analyses and make predictions of future values. Pearson's correlation coefficient is calculated using the following formula:

$$r = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \sqrt{n \sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} y_i)^2}}$$

where

$$
\begin{array}{rcl}
r & \equiv & \text{the correlation coefficient,} \\
x_i & \equiv & \text{the value of the independent variable for the } i\text{th observation,} \\
y_i & \equiv & \text{the value of the dependent variable for the } i\text{th observation, and} \\
n & \equiv & \text{the number of pairs of data points.}
\end{array}
$$

The closer r is to -1 or $+1$, the stronger the association; and the closer it is to zero, the weaker the relationship between the pairs of variables.

Example 8.8

The most important purpose of advertisements is to boost the sales of products. Suppose a company keeps a record of the money spent on advertising in a month and the sales in the following month. This is given in the table below:

Advertising expenses in R100 000 (x)	Sales during the month after the advertisement (y)
10	5
14	8
13	6
15	7
18	10

Before the correlation coefficient can be calculated, the following values must be worked out:

- the sum of each column ($\sum x_i$ and $\sum y_i$),

- the number of data points (n),

- the square and the sum of the squares of each variable ($x_i^2, y_i^2, \sum x_i^2$ and $\sum y_i^2$), and

- the product and the sum of the products of the two variables ($x_i y_i$ and $\sum x_i y_i$).

The example now looks like this (the sum of each column is printed in bold at the bottom of the column):

n	x_i	y_i	x_i^2	y_i^2	$x_i \times y_i$
1	10	5	100	25	50
2	14	8	196	64	112
3	13	6	169	36	78
4	15	7	225	49	105
5	18	10	324	100	180
TOTAL	**70**	**36**	**1 014**	**274**	**525**

In this case, Pearson's correlation coefficient is therefore:

$$
\begin{aligned}
r &= \frac{5 \times 525 - 70 \times 36}{\sqrt{[5 \times 1\,014 - 70^2][5 \times 274 - 36^2]}} \\
&= \frac{2\,625 - 2\,520}{\sqrt{(5\,070 - 4\,900)(1\,370 - 1\,296)}} \\
&= \frac{105}{\sqrt{170 \times 74}} \\
&= 0{,}9362
\end{aligned}
$$

NB: Use your calculator to do these types of calculations. See Tutorial Letter 101 for the key operations.

The value of the correlation coefficient is very close to 1. From this can be deduced that there is a *high positive correlation* between the advertising expenses and the company's sales in the following month.

Exercise 8.3

Calculate the correlation coefficient between *number of years' experience* and *annual sales* for Example 8.7.

8.4.1.3   Coefficient of determination

Although the correlation coefficient is often used in practice, it is more common to use the square of the coefficient. It is then called the coefficient of determination and is usually expressed as a percentage.

The coefficient of determination, R^2, represents the part of the variation in the dependent variable (y) that can be explained by the independent variable, (x).

In Example 8.8 above, the coefficient of determination is 87,64% $(0,9362^2 \times 100)$, which implies that 87,64% of the variations in the sales can be explained by the variations in the advertising expenses. (Remember: this does not mean that a causal relationship exists between the two variables.)

Exercise 8.4

Use the data in Example 8.7 to calculate the coefficient of determination.

8.4.2   Regression

Regression is the method by which the *relationship between two variables is determined* and expressed as an equation, usually with the aim of forecasting the behaviour of one of the variables.

In regression analysis, the function that we use to describe the data, is called the projected regression function. Regression analysis where there is a straight-line relationship between the variables and there is one independent variable, is often called *simple linear regression*. Where two or more independent variables are used to predict the value of the dependent variable, multiple regression can be used. This, however, falls outside the scope of this module.

In this section we limit ourselves to the task of determining the straight line that fits the data in a scatter diagram "best". The purpose of simple regression is to determine a linear relationship between the values of just two random variables. Therefore, we will fit a function with the form

$$\hat{y} = b_0 + b_1 x$$

or

$$y = a + bx$$

where

$\hat{y}$ $\equiv$ is the projected value of the dependent variable,

x $\equiv$ is the value of the independent variable,

b_0 $\equiv$ is the intercept on the y axis (value of $\hat{y}$ when $x = 0$) also called a, and

b_1 $\equiv$ the change in the dependent variable that corresponds with the change in the independent variable (that is the gradient of the line), also called b.

Depending on the aspect of the regression procedure that is emphasised, different books describe such regression lines with different names, for example, the least square line, the line of best fit, the line of Y on X, etc.

The "best" line can now be determined as the line for which the sum of the square distances is a minimum.

In order to understand what regression does, we will consider a scatter diagram on which a straight line has been drawn.
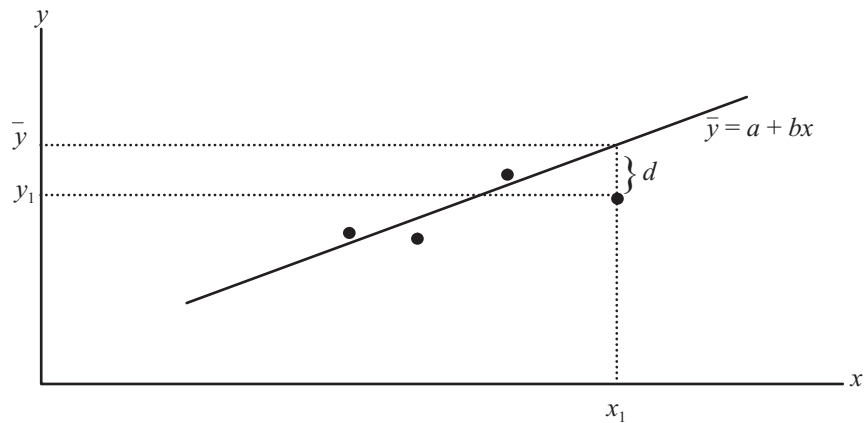
Note that the independent variable is indicated on the horizontal axis (x-axis) and the dependent variable on the vertical axis (y-axis). There is also a vertical distance, d, that is measured from the regression line to the point (x_1, y_1). Such distances can be calculated between the line and all the data points, and are used to get the line that fits the data best.

There are actually an infinite number of straight lines that can be drawn on the scatter diagram, and the vertical distance, d, can be measured for each of them. In an effort to get a "good" line, an aggregate error is calculated that represents (in a certain sense) the total error. The best line is the one that minimises the aggregate error. One possibility is to use the sum of the distances.

However, because some of the d's are positive and some negative, they cancel each other out and can even be zero. In order to overcome this problem, each of the distances is squared and the total (sum) of the squared values is determined. The *"best" line can now be determined as the line for which the sum of the squared distances is a minimum.*

The following formulæ are used to work out the formula of the regression line

$$y = a + bx,$$

where

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

and

$$a = \frac{\sum y_i}{n} - \frac{b \sum x_i}{n}.$$

Therefore, for Example 8.8:

$$b = \frac{105}{170} = 0{,}62$$

and

$$
\begin{aligned}
a &= \frac{36}{5} - \frac{0{,}62 \times 70}{5} \\
&= 7{,}2 - 8{,}68 \\
&= -1{,}48
\end{aligned}
$$

And the equation of the regression line is therefore:

$$y = 0{,}62x - 1{,}48$$

Since $r = 0{,}9362$ and therefore $r^2 = 0{,}8765$, it can be said that the regression model describes $87{,}65\%$ ($r^2 \times 100$) of the variation in the data (which indicates that it is a reasonably good model).

Note: Note that the values of a and b, as well as r, can be calculated very easily with the help of your calculator.

8.5   Summary of Chapter 8

In this chapter the concept of single subscripts and summation as well as double subscripts and summation notation were explained, for example

$$\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3$$

and

$$
\begin{aligned}
\sum_{i=1}^{2} \sum_{j=1}^{3} y_{ij} &= \sum_{i=1}^{2} (y_{i1} + y_{i2} + y_{i3}) \\
&= (y_{11} + y_{12} + y_{13}) + (y_{21} + y_{22} + y_{23}) \\
&= (y_{11} + y_{21}) + (y_{12} + y_{22}) + (y_{13} + y_{23}) \\
&= \sum_{j=1}^{3} (y_{1j} + y_{2j}) \\
&= \sum_{j=1}^{3} \sum_{i=1}^{2} y_{ij}.
\end{aligned}
$$

Two types of measures of the central value of a data set were discussed:

1. The arithmetic mean, which is calculated by the formula

$$
\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}
$$

2. The weighted mean where each data point, x_i has a weight that represents its relative importance

$$
\bar{x}_w = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}
$$

A measure of variation namely the standard deviation was discussed:

$$
S = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}}
$$

The variance is the square of the standard deviation, namely S^2.

The equation describing a linear function is given by $y = a + bx$ where a is the intercept on the y-axis and b is the slope of the line.

The intercept of a straight line on the x-axis is given by the point $(\frac{-a}{b}; 0)$ and on the y-axis by the point $(0; a)$.

From two points P_1 with coordinates $(x_1; y_1)$ and P_2 with coordinates $(x_2; y_2)$ the slope of a straight line can be calculated by

$$
b = \frac{y_2 - y_1}{x_2 - x_1}.
$$

Lastly, correlation and regression were discussed. They are techniques used for the description and analysis of data. Correlation and regression analysis are used to measure the relationship between two variables. Knowledge of this relationship is very useful when making estimates and forecasts. The following concepts were discussed:

Scatter diagrams give a visual indication of the relationship between two variables. A scatter diagram is drawn by plotting the two variables' observations on a graph.

Correlation analysis can be used to determine the strength of the relationship between two variables. When quantitative data are involved, Pearson correlation coefficient is used. Peasons's correlation coefficient is calculated using the following formula:

$$
r = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \sqrt{n \sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} y_i)^2}}
$$

where

r	$\equiv$	the correlation coefficient,
x_i	$\equiv$	the value of the independent variable for the ith observation,
y_i	$\equiv$	the value of the dependent variable for the ith observation, and
n	$\equiv$	the number of pairs of data points.

Although the correlation coefficient is often used in practice, it is more common to use the square of the coefficient. It is then called the coefficient of determination and is usually expressed as a percentage.

The coefficient of determination, R^2, represents the part of the variation in the dependent variable (y) that can be explained by the independent variable, (x).

Regression is the method by which the *relationship between two variables is determined* and expressed as an equation, usually with the aim of forecasting the behavior of one of the variables. In this module we only look at simple linear regression, which means determining the straight line that fits the data in a scatter diagram the "best".

The following formulæ are used to work out the formula of the regression line

Regression line:

$$y = a + bx,$$

where

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

and

$$a = \frac{\sum y_i}{n} - \frac{b \sum x_i}{n}.$$

All the calculations can be done directly on your calculator.

8.6   Evaluation exercises

1. On a certain trip a motorist bought 12 litres of petrol at R10,14 per litre, 18 litres at R10,03 per litre and 50 litres at R10,20 per litre. Find the "average" price this motorist paid per litre of petrol.

2. The following table shows average growth rates (%) of gross income for three major banks in South Africa for four time periods.

Average percentage growth per year

Bank	2007-2008	2008-2009	2009-2010	2010-2011
A	10,9	11,7	24,3	0,2
B	11,6	13,4	27,1	4,2
C	11,5	11,0	13,0	0,1

(a) Compute the standard deviation of percentage growth for each of the banks.

(b) Which bank has the greatest variation in percentage growth? Which bank has the least variation?

3. The relationship between monthly income and years' of employment for employees of a large firm was computed to be:

$$y = 10\,210 + 850x$$

where y is the monthly income in rand and x is the years' of service in the firm.

   (a) What is the estimated income for an employee with five years of service?

   (b) With each year of additional service, how much of a raise in income can an employee expect on the average?

4. The Building Industries Federation were analysing the effect that mortgage interest rates have on the number of building contracts undertaken in the Cape Town area. For 12 quarters randomly selected from the past ten years, average mortgage rates were collected. The number of building contracts completed in the quarter after the chosen mortgage rate quarter was collected (to allow time for mortgage rates to effect building contracts issued).

The data appears in the following table:

Mortgage rate (percentage)	Building contracts undertaken
16,00	443
16,75	448
17,00	367
17,50	492
18,25	384
18,75	437
19,00	356
19,50	339
19,75	365
20,25	321
20,75	338
21,25	230

   (a) Produce a scatter diagram to show the likely relationship between mortgage rates and building contracts completed. Comment on the diagram.

   (b) Find the regression equation $y = a + bx$.

   (c) If mortgage rates decreased from 18% to 16,5% in a given quarter, what effect would this be expected to have on the number of building contracts completed in the next quarter?

   (d) Find the correlation between mortgage rates and building contracts. Comment on the findings.

APPENDIX A

Answers to exercises

> From error to error, one discovers the entire truth.
>
> —Sigmund Freud

Chapter 1. Introduction

Chapter 2. Simple interest and simple discount

Exercise 2.1

Since the interest rate is expressed "per annum", we first have to express the term t in years:

$$t = \frac{90}{365}$$

Then

$$
\begin{aligned}
I &= Prt \\
&= 5\,000 \times 0{,}15 \times \frac{90}{365} \\
&= 184{,}93
\end{aligned}
$$

The interest is R184,93.

$$
\begin{aligned}
S &= P + I \\
&= 5\,000 + 184{,}93 \\
&= 5\,184{,}93
\end{aligned}
$$

The accumulated amount is R5 184,93.

OR

$$
\begin{aligned}
S &= P(1 + rt) \\
&= 5\,000(1 + 0{,}15 \times \frac{90}{365}) \\
&= 5\,184{,}93
\end{aligned}
$$

The accumulated amount is R5 184,93.

Note, as pointed out earlier, reference is occasionally made to a 360-day year. This has its origin in precalculator days when sums of the above type were tedious. In the above example, the effect of this would have been that $t = \frac{90}{360} = \frac{1}{4}$, which obviously makes manual calculation a lot easier. However, unless the contrary is stated, you should always assume that the "exact" year (365 days in a year) is used.

Exercise 2.2
Now $S = \text{R}12\,000$, $r = 0{,}12$ and $t = \frac{3}{12} = \frac{1}{4}$.
Thus

$$
\begin{aligned}
P &= \frac{S}{1 + rt} \\
&= \frac{12\,000}{1 + 0{,}12 \times \frac{1}{4}} \\
&= 11\,650{,}49
\end{aligned}
$$

The present value of the loan when it is paid, is R11 650,49.

Exercise 2.3:1
We have $S = \text{R}4\,000$, $d = 0{,}18$ and $t = \frac{6}{12} = \frac{1}{2}$.

$$
\begin{aligned}
D &= Sdt \\
&= 4\,000 \times 0{,}18 \times \frac{1}{2} \\
&= 360
\end{aligned}
$$

The discount is R360. The discounted value is

$$
\begin{aligned}
P &= S - D \\
&= 4\,000 - 360 \\
&= 3\,640
\end{aligned}
$$

You will therefore receive R3 640.

Since the interest (I) paid is R360, we use

$$
I = Prt.
$$

Therefore

$$
360 = 3\,640 \times r \times \frac{1}{2}
$$

or

$$r = \frac{360}{3\,640} \times 2$$

$$= 0{,}1978.$$

Thus the equivalent simple interest rate is 19,78% per annum.

Exercise 2.3:2

(a) We take $S = 100$. Since $d = 0{,}12$ and $t = \frac{3}{12} = \frac{1}{4}$ we find

$$D = Sdt$$

$$= 100 \times 0{,}12 \times \tfrac{1}{4}$$

$$= 3$$

and $\quad P = S - D$

$$= 100 - 3$$

$$= 97.$$

Since $I = 3$ and $\quad I = Prt$

$$3 = 97 \times r \times \tfrac{1}{4}$$

thus $\quad r = \tfrac{3}{97} \times 4$

$$= 0{,}1237.$$

OR $\quad S = P(1 + rt)$

$$P = S(1 - dt)$$

thus $\quad S = S(1 + rt)(1 - dt)$

$$\tfrac{1}{1-dt} = 1 + rt$$

$$rt = \tfrac{1}{1-dt} - 1$$

$$= \tfrac{1-1+dt}{1-dt}$$

$$= \tfrac{dt}{1-dt}$$

$$r = \tfrac{dt}{1-dt} \times \tfrac{1}{t}$$

$$= \tfrac{d}{1-dt}$$

Substitute the given values in the formula

$$r = \frac{0,12}{1-0,12\times\frac{3}{12}}$$

$$= 0,1237.$$

Thus the equivalent simple interest rate is 12,37% per annum.

(b) Again $S = $ R100 and $d = 0,12$ but now $t = \frac{9}{12} = \frac{3}{4}$. Thus

$$D = 100 \times 0,12 \times \frac{3}{4}$$

$$= 9$$

and

$$P = 100 - 9$$

$$= 91.$$

From

$$I = Prt$$

$$9 = 91 \times r \times \frac{3}{4}$$

thus

$$r = \frac{4}{3} \times \frac{9}{91}$$

$$= 0,1319.$$

OR

$$r = \frac{d}{1-dt}$$

$$= \frac{0,12}{1-0,12\times\frac{3}{4}}$$

$$= 0,1319.$$

In this case, the equivalent simple interest rate is 13,19% per annum.

Exercise 2.4

Count and add the days from 12 October to 15 May of the following year as follows:

Month	Number of days
October	20 (including 12 October)
November	30
December	31
January	31
February	28
March	31
April	30
May	14 (excluding 15 May)
	$\overline{215}$

OR

Day number 365 (31 December) minus day number 285 (12 October) plus day number 135 (15 May) equals 215.

There are thus 215 days between the two dates.

Exercise 2.5
First determine the day number of the maturity date, namely 2 July; day number 183; and then subtract the day number of the settlement date 13 May; day number 133. Thus $183 - 133 = 50$ days. The value on the maturity date is the so-called "par value", which is given as R1 000 000. The discount rate is given as 16,55% per annum.

$$
\begin{aligned}
P &= S(1 - dt) \\
&= 1\,000\,000 \left(1 - 0{,}1655 \times \frac{50}{365} \right) \\
&= 977\,328{,}77
\end{aligned}
$$

The present value on the settlement date is therefore R977 328,77. This is the price the investor must pay on 13 May in order to receive R1 000 000 on 2 July (ie 50 days later).

Now

$$
\begin{aligned}
I &= 1\,000\,000 - 977\,328{,}77 \\
&= 22\,671{,}23
\end{aligned}
$$

The interest earned is therefore R22 671,23. Since such interest is on the purchase price, we can calculate the equivalent simple interest rate as follows:

$$
\begin{aligned}
I &= Prt \\
r &= \frac{I}{Pt} \\
&= \frac{22\,671{,}23}{977\,328{,}77 \times \frac{50}{365}} \\
&= 0{,}1693391
\end{aligned}
$$

OR

$$
\begin{aligned}
r &= \frac{d}{1 - dt} \\
&= \frac{0{,}1655}{1 - 0{,}1655 \times \frac{50}{365}} \\
&= 16{,}93\%
\end{aligned}
$$

Thus the equivalent simple interest rate is 16,93% per annum.

Exercise 2.6
The three problems can be represented on a time line (all at 15% per annum):

(a)
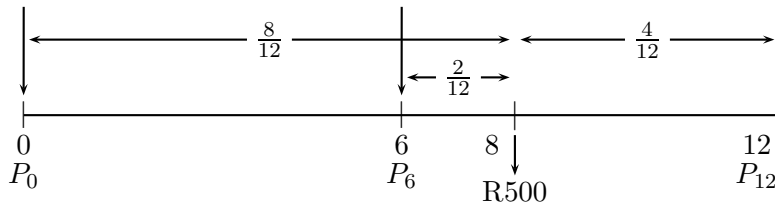
$$P_0 = \frac{500}{1 + 0.15 \times \frac{8}{12}}$$

$$= 454.55$$

Melanie must pay R454,55 now.

(b)

$$P_6 = \frac{500}{1 + 0.15 \times \frac{2}{12}}$$

$$= 487.80$$

Melanie must pay R487,80 at month six.

(c)

$$P_{12} = 500 \times \left(1 + 0.15 \times \frac{4}{12}\right)$$

$$= 525.00$$

Melanie must pay R525,00 at month 12.

Exercise 2.7
In the following diagram, debts are shown above the line and payments below.
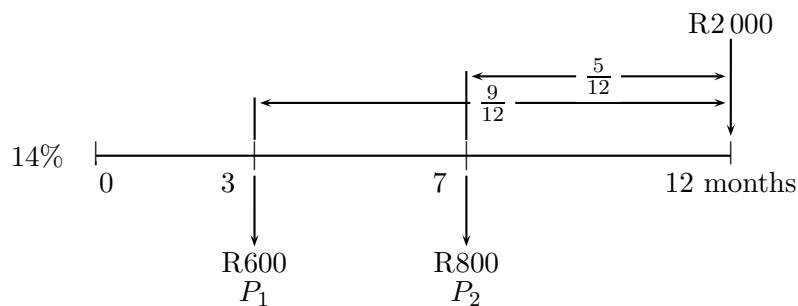
The values of the two payments at the end of the year are respectively:

$$P_1 = 600 \times \left(1 + 0.14 \times \frac{9}{12}\right)$$

$$= 663.00$$

154

$$P_2 = 800 \times \left(1 + 0{,}14 \times \frac{5}{12}\right)$$
$$= 846{,}67$$

Thus, at the end of the year, Mary-Jane has effectively already paid off R1 509,67 (663,00 + 846,67).

She must thus pay R490,33 (2 000,00 − 1 509,67) to settle her debt.

Chapter 3. Compound interest and equations of value

Exercise 3.1:1

$$S = P(1 + i)^n.$$

Use $P = 5\,000$, $i = 0{,}075$ (ie $7\frac{1}{2} \div 100$) and $n = 10$. Now

$$S = 5\,000(1 + 0{,}075)^{10}$$
$$= 5\,000(1{,}075)^{10}$$
$$= 10\,305{,}16.$$

The compounded amount (or accrued principal) R10 305,16.

Exercise 3.1:2

$$S = P(1 + i)^n.$$

Use $P = 9\,000$, $i = 0{,}08$ (ie $8 \div 100$) and $n = 5$. Now

$$S = 9\,000(1 + 0{,}08)^5$$
$$= 9\,000(1{,}08)^5$$
$$= 13\,223{,}95.$$

The accumulated amount is R13 223,95.

The interest earned is R4 223,95 (13 223,95 − 9 000).

Exercise 3.2
$P = 2\,000$. The formula to use is $S = P\left(1 + \frac{j_m}{m}\right)^{tm}$.

(a) If the interest is calculated yearly, $j_m = 0{,}14$, $m = 1$ and $t = 3$. Then

$$S = P\left(1 + \frac{j_m}{m}\right)^{tm}$$
$$= 2\,000(1 + 0{,}14)^3$$
$$= 2\,963{,}09.$$

The accumulated amount is R2 963,09.
The applicable time line is:

R2 000

14%

0	1	2	3

R2 963,09
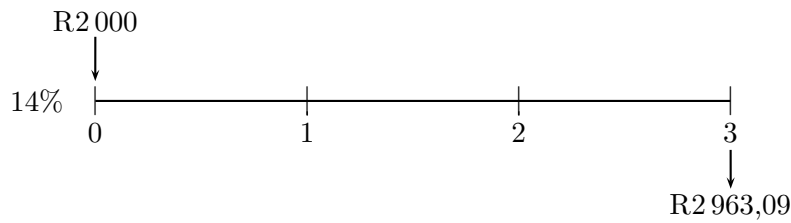
(b) If the interest is calculated half-yearly, $j_m = 0{,}14$, $m = 2$ and $t = 3$. Then

$$
\begin{aligned}
S &= P\left(1 + \frac{j_m}{m}\right)^{tm} \\
&= 2\,000\left(1 + \frac{0{,}14}{2}\right)^{(3\times2)} \\
&= 3\,001{,}46.
\end{aligned}
$$

The accumulated amount is R3 001,46.

The applicable time line is:

R2 000

7%

0	1	2	3	4	5	6

R3 001,46

(c) If the interest is calculated quarterly, $j_m = 0{,}14$, $m = 4$ and $t = 3$. Then

$$
\begin{aligned}
S &= P\left(1 + \frac{j_m}{m}\right)^{tm} \\
S &= 2\,000\left(1 + \frac{0{,}14}{4}\right)^{(3\times4)} \\
&= 3\,022{,}14.
\end{aligned}
$$

The accumulated amount is R3 022,14.

The applicable time line is:

R2 000

3,5%

0	1		11	12

R3 022,14
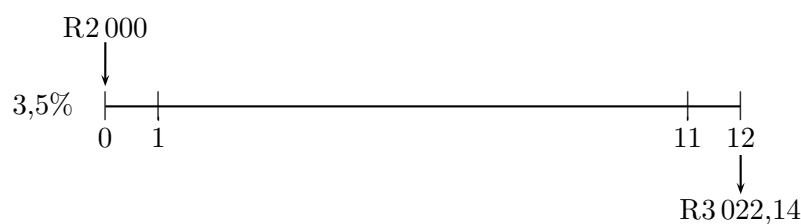
(d) If the interest is calculated daily, $j_m = 0{,}14$ $m = 365$ and $t = 3$. Then

$$
\begin{aligned}
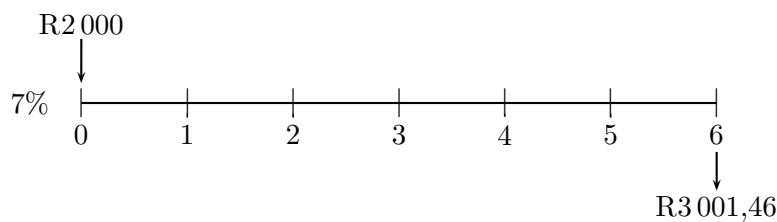S &= P\left(1 + \frac{j_m}{m}\right)^{tm} \\
S &= 2\,000\left(1 + \frac{0{,}14}{365}\right)^{(365\times 3)} \\
&= 3\,043{,}68.
\end{aligned}
$$

The accumulated amount is R3 043,68.

The applicable time line is:
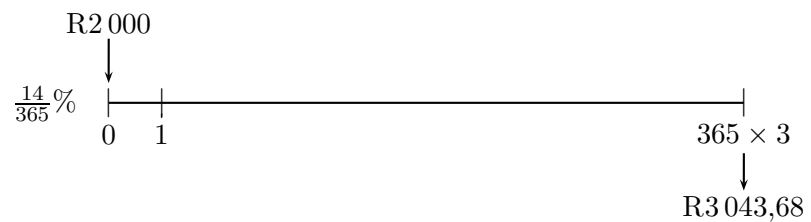
Exercise 3.3

$$
P = S\left(1 + \frac{j_m}{m}\right)^{-tm}
$$

with $j_m = 0{,}18$, $m = 12$ and $t = 5$.

Then

$$
\begin{aligned}
P &= 10\,000\left(1 + \frac{0{,}18}{12}\right)^{-(5\times 12)} \\
&= 4\,092{,}96.
\end{aligned}
$$

That is, R4 092,96 must be invested now at 18% per annum compounded monthly, in order to accrue to R10 000 in five years' time. The relevant time line is as follows:

Exercise 3.4

Now we have $P = $ R2 500, $S = 2 \times 2\,500 = $ R5 000, $j_m = 0{,}11$ and $m = 2$.

Then

$$t = \frac{\ln\left(\frac{S}{P}\right)}{m \ln\left(1 + \frac{j_m}{m}\right)}$$

$$= \frac{\ln\left(\frac{5\,000}{2\,500}\right)}{2 \ln\left(1 + \frac{0,11}{2}\right)}$$

$$= 6,47.$$

The time under consideration is 6,47 years. (Remember t is defined as time in years.)

Note that this result holds for any "doubling", that is, irrespective of the actual amounts.

Of course, your calculator can give this result in one step.

Exercise 3.5

Suppose an amount of P is invested. Then the accrued amount S must equal $3P$.

$$S = P\left(1 + \frac{j_m}{m}\right)^{tm}$$

with $\frac{S}{P} = 3$, $m = 1$ and $t = 10$.

$$j_m = m\left(\left(\frac{S}{P}\right)^{\frac{1}{tm}} - 1\right)$$

$$= 1\left(3^{\frac{1}{1 \times 10}} - 1\right)$$

$$= 3^{\frac{1}{10}} - 1$$

$$= 0,1161$$

The required interest rate is 11,61% per annum.

Exercise 3.6

$$J_{eff} = 100\left(\left(1 + \frac{j_m}{m}\right)^m - 1\right).$$

Using the formula with $j_m = 0,22$ and $m = 2, 4, 12$ and 365 respectively, we find the following effective interest rates expressed as a percentage:

(a) half-yearly: 23,21% per annum (23,21000%)

(b) quarterly: 23,88% per annum (23,88247%)

(c) monthly: 24,36% per annum (24,35966%)

(d) daily: 24,60% per annum (24,59941%)

Exercise 3.7

The applicable time line is:

R10 000

| | | |
15 March 2010   1 July 2010   1 July 2012

simple interest

compound interest
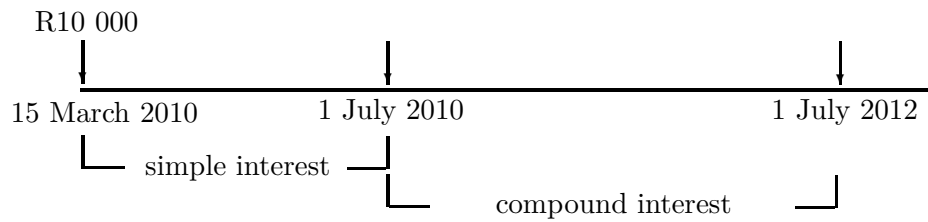
The calculation must be performed in two parts:

(a) 15 March 2010 to 1 July 2010

108 days (check!) at $15\frac{1}{2}\%$ per annum simple interest.

$$\begin{aligned}
S &= P(1 + rt) \\
&= 10\,000(1 + 0{,}155 \times 108 \div 365) \\
&= 10\,458{,}63
\end{aligned}$$

The sum accumulated is R10 458,63.

(b) This R10 458,63 must be invested for two years at $15\frac{1}{2}\%$ per annum compound interest.

$$\begin{aligned}
S &= P\left(1 + \frac{j_m}{m}\right)^{tm} \\
&= 10\,458{,}63 \times (1 + 0{,}155)^2 \\
&= 13\,952{,}07
\end{aligned}$$

The sum accumulated is R13 952,07.

Thus, over the term of two years and 108 days, the R10 000 earns R3 952,07 in interest.

Exercise 3.8

(a) Simple interest is used and then followed by compound interest:

$$\begin{aligned}
S &= P(1 + rt_1)\left(1 + \frac{j_m}{m}\right)^{t_2 m} \qquad (t_1 = 3 \div 12;\ t_2 = 5;\ m = 1) \\
&= 5\,000(1 + 0{,}18 \times 3 \div 12)(1 + 0{,}18)^{5 \times 1} \\
&= 5\,225(1 + 0{,}18)^{5 \times 1} \\
&= 11\,953{,}53
\end{aligned}$$

The sum accumulated is R11 953,53.

(b) For fractional compounding, the sum accumulated is

$$
\begin{aligned}
S &= P\left(1 + \frac{j_m}{m}\right)^{tm} \qquad (t = 5 + 3 \div 12,\ m = 1) \\
&= 5\,000 \times (1 + 0{,}18)^{(5+0,25)} \\
&= 11\,922{,}04
\end{aligned}
$$

The sum accumulated is R11 922,04.

Exercise 3.9

$$
\begin{aligned}
J_\infty &= 100(e^c - 1) \\
&= 100(e^{0,22} - 1) \\
&= 24{,}608\%
\end{aligned}
$$

The effective rate is 24,608%.

In exercise 3.6 above, the effective rate is 24,59941% for daily compounding.

Exercise 3.10
The number of days from 15 November to 18 May of the following year is 184. Day number 365 minus day number 319 (15 November) plus day number 138 (18 May). Thus the fraction of the year for which the money is invested is $\frac{184}{365}$.

$$
\begin{aligned}
S &= Pe^{ct} \\
&= 12\,000e^{(0,16 \times 184 \div 365)} \\
&= 13\,008{,}00
\end{aligned}
$$

The sum accumulated is R13 008,00.

Exercise 3.11
We use the formula $c = m\ln(1 + \frac{j_m}{m})$ to calculate the continuous rate in each case:

(a) $c = 2\ln(1 + \frac{0,1975}{2}) = 18{,}835\%$

(b) $c = 12\ln(1 + \frac{0,19}{12}) = 18{,}851\%$

The second option is marginally better than the first.

Exercise 3.12
This is really a straightforward present value calculation, namely

$$
P = S\left(1 + \frac{j_m}{m}\right)^{-(tm)}
$$

where $S = 120\,000$, $j_m = 0{,}15$, $m = 12$ and $t = 4$.

$$
\begin{aligned}
P &= 120\,000\left(1 + \frac{0,15}{12}\right)^{-(4 \times 12)} \\
&= 66\,102{,}78
\end{aligned}
$$

You must deposit R66 102,78 now – and then study hard for your degree in order to qualify, so that you can buy your car!

Exercise 3.13

We must first calculate the maturity values of her *obligations*.

On R4 000, with $j_m = 0{,}12$, $m = 12$ and $t = 5$.

$$
\begin{aligned}
S &= 4\,000 \times \left(1 + \frac{0{,}12}{12}\right)^{(5 \times 12)} \\
&= 7\,266{,}79
\end{aligned}
$$

The R4 000 accumulated to R7 266,79 and is due two years from now.

On R8 000, with $m = 4$, $j_m = 0{,}16$ and $t = 5$:

$$
\begin{aligned}
S &= 8\,000 \times \left(1 + \frac{0{,}16}{4}\right)^{5 \times 4} \\
&= 17\,528{,}99
\end{aligned}
$$

The R8 000 accumulated to R17 528,99 and is due four years from now.

We denote each payment by X. The obligations and payments are indicated on the following time line:
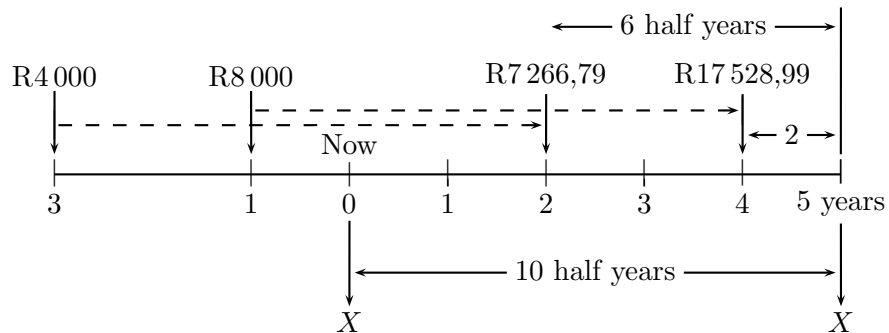
The interest rate is $j_m = 0{,}2$ and t is the number of years' for each obligation and payment.

$$
\begin{aligned}
\text{Payments} &= \text{Obligations} \\
X \times \left(1 + \frac{0{,}2}{2}\right)^{5 \times 2} + X &= 7\,266{,}79 \left(1 + \frac{0{,}2}{2}\right)^{3 \times 2} + 17\,528{,}99 \left(1 + \frac{0{,}2}{2}\right)^{1 \times 2} \\
X \times (1 + 0{,}1)^{10} + X &= 7\,266{,}79 \, (1 + 0{,}1)^6 + 17\,528{,}99 \, (1 + 0{,}1)^2 \, .
\end{aligned}
$$

Take the X out as the common term

$$
X(1{,}1^{10} + 1) = 7\,266{,}79 \times 1{,}1^6 + 17\,528{,}99 \times 1{,}1^2 \, .
$$

$$
\begin{aligned}
X &= \frac{\left(7\,266{,}79 \times 1{,}1^6 + 17\,528{,}99 \times 1{,}1^2\right)}{(1{,}1^{10} + 1)} \\
&= 9\,484{,}16
\end{aligned}
$$

Therefore each payment, one now, and one five years hence, is R9 484,16.

Chapter 4.　Annuities

Exercise 4.1

Following the pattern of Example 4.1, the relevant time line is:
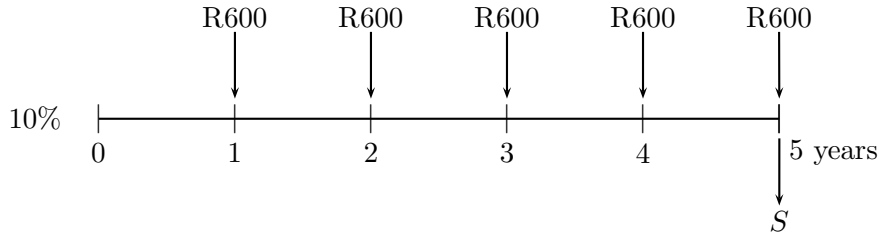
$$
\begin{aligned}
S &= S_1 + S_2 + S_3 + S_4 + S_5 \\
&= 600(1+0{,}1)^4 + 600(1+0{,}1)^3 + 600(1+0{,}1)^2 + 600(1+0{,}1)^1 + 600 \\
&= 3\,663{,}06
\end{aligned}
$$

The accumulated amount is R3 663,06.

Exercise 4.2:1

$$
S = Rs_{\overline{n}|i}
$$

$R = 600$, $n = 5$ and $i = 0{,}10$.

$$
\begin{aligned}
S &= 600 s_{\overline{5}|0,10} \\
&= 3\,663{,}06
\end{aligned}
$$

The accumulated amount is R3 663,06.

Exercise 4.2:2

The relevant time line is:
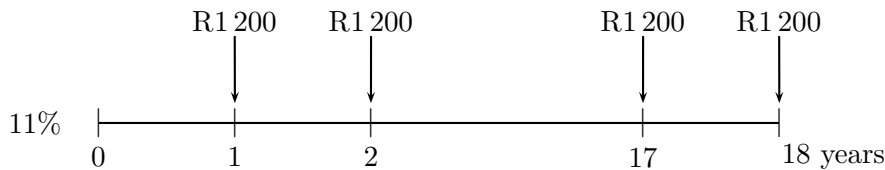
The payment is thus $R = 1\,200$ per year, whereas the interest rate is $i = 0{,}11$ per year, and the number of intervals is $n = 18$.

$$
\begin{aligned}
S &= 1\,200 s_{\overline{18}|0,11} \\
&= 1\,200 \left(\frac{1{,}11^{18} - 1}{0{,}11} \right) \\
&= 60\,475{,}12
\end{aligned}
$$

When her daughter is 18 years old, the accumulated amount will be R60 475,12.

Exercise 4.2:3
The time line is:
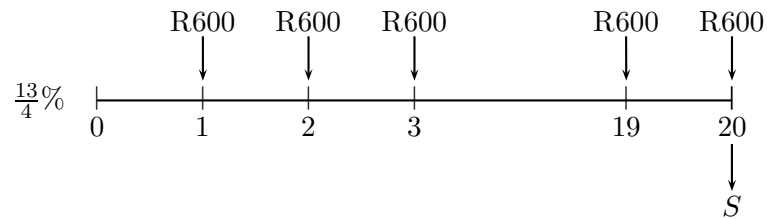
Now $R = 600$, $i = \frac{0,13}{4} = 0,0325$ and $n = 5 \times 4 = 20$.

$$
\begin{aligned}
S &= 600 s_{\overline{5 \times 4}|0,13 \div 4} \\
&= 600 \left(\frac{1,0325^{20} - 1}{0,0325} \right) \\
&= 16\,538,55
\end{aligned}
$$

The accumulated amount at the end of the term is R16 538,55.

Exercise 4.3:1

$$
P = R a_{\overline{n}|i}.
$$

$R = 1\,000$, $n = 5$ and $i = 0,125$.

$$
\begin{aligned}
P &= 1\,000 a_{\overline{5}|0,125} \\
&= 3\,560,57
\end{aligned}
$$

The present value is R3 560,57.

Exercise 4.3:2
The relevant time line is:
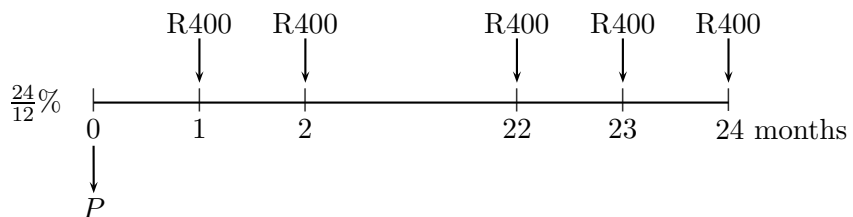
and $i = \frac{0,24}{12}$.

The present value of the 24 payments is

$$
\begin{aligned}
P &= 400 a_{\overline{24}|0,24 \div 12} \\
&= 7\,565,57.
\end{aligned}
$$

The car costs R10 565,57 (3 000+7 565,57). The interest paid is the difference between the sum of all payments and the cost, namely

$$
\begin{aligned}
I &= (3\,000 + 24 \times 400) - 10\,565,57 \\
&= 12\,600 - 10\,565,57 \\
&= 2\,034,43.
\end{aligned}
$$

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The interest paid is R2 034,43.

Exercise 4.3:3

The relevant time line is:
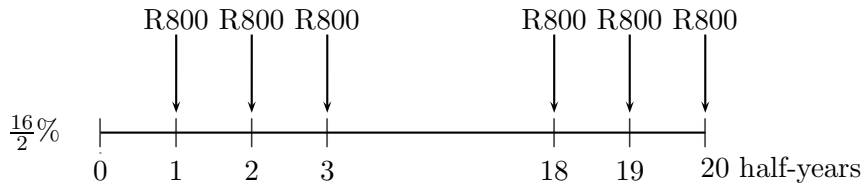
and $i = \frac{0,16}{2}$, $n = 2 \times 10 = 20$.

$$
\begin{aligned}
P &= 800a_{\overline{10 \times 2}|0,16 \div 2} \\
&= 800 \left(\frac{1,08^{20} - 1}{0,08 \times 1,08^{20}} \right) \\
&= 7\,854,52
\end{aligned}
$$

The present value is R7 854,52.

This is the amount that must be invested now at 16% per annum, compounded half-yearly, to provide half-yearly payments of R800 for ten years.

Exercise 4.4

The time line is:
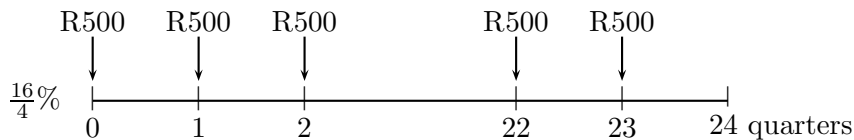
The interest rate per quarter is $16\% \div 4$, and the number of intervals $6 \times 4 = 24$.

Since the first payment is made immediately, and the remaining 23 form an annuity certain, the present value is:

$$
\begin{aligned}
P &= 500 + 500a_{\overline{23}|0,16 \div 4} \\
&= 500 + 7\,428,42 \\
&= 7\,928,42
\end{aligned}
$$

Exercise 4.5

As you saw, we can write the present value, P for an annuity due, with payments of R at an interest rate per period of i for n periods, as follows:

$$
P = R + Ra_{\overline{n-1}|i}.
$$

The first term represents the first payment, whereas the second term represents the present value of an ordinary annuity with $n - 1$ periods.

Now all we need is a bit of mathematical sleight of hand. See if you can follow me – I'll take it slowly.

$$P \quad = R + Ra_{\overline{n-1}|i}$$

$$= R\left(1 + a_{\overline{n-1}|i}\right) \qquad \text{(Take out the common factor } R\text{)}$$

$$= R\left(1 + \frac{(1+i)^{n-1}-1}{i(1+i)^{n-1}}\right)$$

$$= R\left(\frac{i(1+i)^{n-1}+(1+i)^{n-1}-1}{i(1+i)^{n-1}}\right)$$

$$= R\left(\frac{(1+i)^{n-1}(i+1)-1}{i(1+i)^{n-1}}\right) \qquad \text{(Group the first two terms)}$$

$$= R\left(\frac{(1+i)^{n}(i+1)^{-1}(i+1)^{1}-1}{i(1+i)^{n-1}}\right) \qquad \text{(Apply the exponential law } a^{m}a^{n}=a^{m+n}\text{)}$$

$$= R\left(\frac{(1+i)^{n}-1}{i(1+i)^{n}(1+i)^{-1}}\right) \qquad \text{(Simplify)}$$

$$= R\left(\frac{((1+i)^{n}-1)(1+i)^{1}}{i(1+i)^{n}}\right) \qquad \text{(Apply } \frac{1}{a^{-m}}=a^{m}\text{)}$$

$$= (1+i)R\frac{(1+i)^{n}-1}{i(1+i)^{n}} \qquad \text{(Simplify)}$$

$$= (1+i)Ra_{\overline{n}|i}$$

Quite straightforward if you know where you are going!

The amount, or sum accumulated, is the future value of P. Thus:

$$\begin{aligned} S \quad &= \quad (1+i)^{n}P \\ &= \quad (1+i)^{n}(1+i)Ra_{\overline{n}|i} \\ &= \quad (1+i)R(1+i)^{n}a_{\overline{n}|i} \end{aligned}$$

But

$$(1+i)^{n}a_{\overline{n}|i} = s_{\overline{n}|i}.$$

Thus

$$S = (1+i)Rs_{\overline{n}|i}.$$

Exercise 4.6
The time line is:
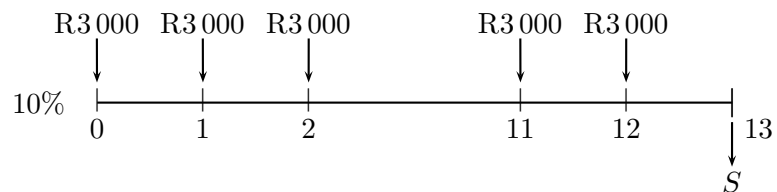
$$\begin{aligned} S \quad &= \quad (1+i)Rs_{\overline{n}|i} \\ &= \quad (1+0{,}1)Rs_{\overline{12}|0,1} \\ &= \quad 70\,568{,}14 \end{aligned}$$

The value at the end of the 12th year is R70 568,14.

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Exercise 4.7

$$
\begin{aligned}
A &= 1\,200 \times 20 + 200 \times 19 + 200 \times 18 + \\
&\quad \cdots + 200 \times 2 + 200 \\
&= 1\,200 \times 20 + 200(19 + 18 + \cdots + 1) \\
&= 1\,200 \times 20 + 200 \left(\frac{19 \times 20}{2} \right) \\
&= 1\,200 \times 20 + 200 \times 190 \\
&= 62\,000
\end{aligned}
$$

OR

$$
\begin{aligned}
A &= Rn^* + Q \left(\frac{n(n-1)}{2} \right) \\
&= 1\,200 \times 20 + 200 \left(\frac{20(20-1)}{2} \right) \\
&= 24\,000 + 38\,000 \\
&= 62\,000
\end{aligned}
$$

The actual amount paid is R62 000.

$$
\begin{aligned}
I &= S - A \\
&= 232\,857{,}08 - 62\,000 \\
&= 170\,857{,}08
\end{aligned}
$$

The interest received is R170 857,08.

* (where n is the number of payments made.)

Chapter 5. Amortisation and sinking funds

Exercise 5.1:1

The present value of the loan is R135 000 (ie $180\,000 - 45\,000$). The number of payments is 12×20, while the interest rate is $18\% \div 12$.

$$
P = R a_{\overline{m}|i}
$$

$$
R = \frac{135\,000}{a_{\overline{20 \times 12}|0{,}18 \div 12}}
$$

$$
= 2\,083{,}47
$$

The monthly payments are R2 083,47.

Exercise 5.1:2

The present value is R60 000, the number of payments 10, and the interest rate 12%.

$$P = Ra_{\overline{n}|i}$$

$$R = \frac{60\,000}{a_{\overline{10}|0,12}}$$

$$= 10\,619{,}05$$

You will receive R10 619,05 at the end of each year for 10 years.

Exercise 5.2:1

There are 3×2 payments at $12\% \div 2$

$$R = \frac{P}{a_{\overline{n}|i}}$$

$$R = \frac{4\,000}{a_{\overline{3 \times 2}|0,12 \div 2}}$$

$$= 813{,}45$$

The half-yearly payments are R813,45.

The amortisation schedule is as follows:

Period (ie half-years)	Outstanding principal at period beginning	Interest due at period end	Payment	Principal repaid	Principal at end of period
1	4 000,00	240,00	813,45	573,45	3 426,55
2	3 426,55	205,59	813,45	607,86	2 818,69
3	2 818,69	169,12	813,45	644,33	2 174,36
4	2 174,36	130,46	813,45	682,99	1 491,38
5	1 491,38	89,48	813,45	723,97	767,41
6	767,41	46,04	813,45	767,41	0,00
Total		880,69	4 880,70	4 000,01	

Exercise 5.2:2

In exercise 5.1(1) the monthly payment on the bond was calculated as R2 083,47.

Interest due at the end of month 1:

$$
\begin{aligned}
I &= S - P \\
  &= P\left(1 + \frac{j_m}{m}\right)^{tm} - P \\
  &= 135\,000\left(1 + \frac{0{,}18}{12}\right)^{\left(\frac{1}{12} \times \frac{12}{1}\right)} - 135\,000 \\
  &= 2\,025{,}00
\end{aligned}
$$

$$\text{OR}$$

$$
\begin{aligned}
I &= Prt \\
  &= 135\,000 \times 0{,}18 \times \frac{1}{12} \\
  &= 2\,025{,}00
\end{aligned}
$$

The interest paid is R2 025,00.

The amortisation schedule for the first six payments is as follows:

Month	Outstanding principal at the beginning of the month	Interest due at the end of the month	Payment	Principal repaid	Principal at month end
1	135 000,00	2 025,00	2 083,47	58,47	134 941,53
2	134 941,53	2 024,12	2 083,47	59,35	134 882,18
3	134 882,18	2 023,23	2 083,47	60,24	134 821,94
4	134 821,94	2 022,33	2 083,47	61,14	134 760,80
5	134 760,80	2 021,41	2 083,47	62,06	134 698,74
6	134 698,74	2 020,48	2 083,47	62,99	134 635,75

Note

At the beginning of the 7th month (after 0,5 years) the outstanding principal will be R134 635,75 ($134\,698{,}74 - 62{,}99$). Let's check whether this agrees with my previous statement that this equals the present value of the payments still to be made:

$$
\begin{aligned}
P &= 2\,083{,}47\,a_{\overline{(20-0{,}5)\times12}|\,0{,}18\div12} \\
  &= 134\,635{,}72
\end{aligned}
$$

The outstanding amount at the beginning of month seven is R134 635,72.

This is close enough! The three cents difference is caused by rounding errors because we work with only two decimals in the amortisation schedule.

To draw up the amortisation schedule for the last six payments on the loan, we must first determine the value of the outstanding principal at the beginning of the 235th month (after 19,5 years).

$$
\begin{aligned}
P &= 2\,083{,}47\,a_{\overline{(20-19{,}5)\times12}|\,0{,}18\div12} \\
  &= 11\,869{,}92
\end{aligned}
$$

The present value of the remaining six payments is R11 869,92.

The amortisation schedule is:

Month	Outstanding principal at the beginning of the month	Interest due at the end of the month	Payment	Principal repaid	Principal at month end
235	11 869,92	178,05	2 083,47	1 905,42	9 964,50
236	9 964,50	149,47	2 083,47	1 934,00	8 030,50
237	8 030,50	120,46	2 083,47	1 963,01	6 067,49
238	6 067,49	91,01	2 083,47	1 992,46	4 075,03
239	4 075,03	61,13	2 083,47	2 022,34	2 052,69
240	2 052,69	30,79	2 083,47	2 052,68	0,01

Exercise 5.3
We saw in exercise 5.1(1) that the monthly repayments are $R = $ R2 083,47. Thus after 12 years there are 8 years to go. At that stage the outstanding principal is the present value of the remaining 96 payments. That is

$$
\begin{aligned}
P &= 2\,083{,}47 a_{\overline{(20-12)\times 12}|0,18\div 12} \\
&= 105\,635{,}42
\end{aligned}
$$

The present value is R105 635,42.

This is, as we have said, the outstanding principal, or what is owed on the initial loan of R135 000. Thus, you have paid off R29 364,58 $(135\,000 - 105\,635{,}42)$. Adding the down payment of R45 000, we obtain your equity, namely R74 364,58.

Exercise 5.4
If the semi-annual deposit into the sinking fund is R, then, after six years at 12% per annum, the fund should have accumulated to R20 000.

$$
\begin{aligned}
S &= R s_{\overline{n}|i} \\
R &= \frac{20\,000}{s_{\overline{6\times 2}|0,12\div 2}} \\
&= 1\,185{,}54
\end{aligned}
$$

The semi-annual deposits are R1 185,54.

The interest that must be paid each quarter is 4% of R20 000, that is, R800

$$
\begin{aligned}
\text{Total yearly cost} &= 4 \times 800 + 2 \times 1\,185{,}54 \\
&= 3\,200 + 2\,371{,}08 \\
&= 5\,571{,}08
\end{aligned}
$$

The total yearly cost (C) for servicing the debt is R5 571,08.

Chapter 6.　Evaluation of cash flows

Exercise 6.1

Now $C = $ R16 500, $I_{out} = $ R10 000 and $n = 4$. Thus

$$
\begin{aligned}
i &= \left(\frac{C}{I_{uit}}\right)^{\frac{1}{n}} - 1 \\
&= \left(\frac{16\,500}{10\,000}\right)^{\left(\frac{1}{4}\right)} - 1 \\
&= 0{,}1334.
\end{aligned}
$$

Thus the internal rate of return is 13,34% per annum.

Exercise 6.2

Now $I_{out} = $ R54 000 and $C = $ R9 500 so that

$$
\frac{I_{out}}{C} = \frac{540}{95}
$$

and

$$
f(i) = \left(\frac{(1+i)^8 - 1}{i(1+i)^8}\right) - \frac{540}{95}.
$$

OR

$$
\begin{aligned}
P &= Ra_{\overline{n}|i} \\
540 &= 95a_{\overline{8}|i} \\
i &= 0{,}083.
\end{aligned}
$$

Using your calculator, you will find that $i = 0{,}083$.

Thus the internal rate of return is 8,3%.

Exercise 6.3

The time line is:
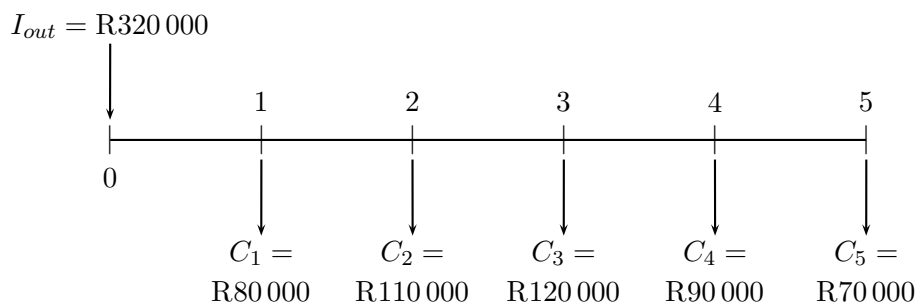
Thus

$$
f(i) = \frac{80\,000}{(1+i)} + \frac{110\,000}{(1+i)^2} + \frac{120\,000}{(1+i)^3} + \frac{90\,000}{(1+i)^4} + \frac{70\,000}{(1+i)^5} - 320\,000.
$$

Using your calculator, you will find $i = 0{,}147$.

Thus the internal rate of return is 14,7% per annum.

Exercise 6.4

The time line for the cash flows is

$I_{out} = $ R800 000
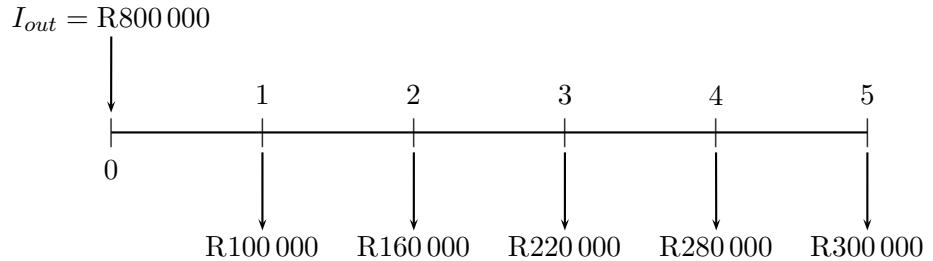
The net present value of the series of cash inflows is

$$\text{NPV} = \frac{100\,000}{1+K} + \frac{160\,000}{(1+K)^2} + \frac{220\,000}{(1+K)^3} + \frac{280\,000}{(1+K)^4} + \frac{300\,000}{(1+K)^5} - 800\,000$$

where K is the cost of capital.

For $K = 9\%$ we find

$$\text{NPV} = -10\,369 < 0.$$

For $K = 8\%$ we find

$$\text{NPV} = 14\,393 > 0.$$

Thus, if the cost of capital is 9%, the investor should not proceed; but if it is 8%, he could consider the investment.

Exercise 6.5

Proposal A:

$$\begin{aligned} \text{NPV} &= \frac{85\,000}{(1+0{,}12)} + \frac{65\,000}{(1+0{,}12)^2} - 100\,000 \\ &= 27\,710 > 0 \end{aligned}$$

The net present value of proposal A's cash inflows is R27 710.

$$\begin{aligned} PI &= \frac{\text{NPV} + \text{ outlays}}{\text{outlays}} \\ PI &= \frac{127\,710}{100\,000} = 1{,}277 > 1. \end{aligned}$$

The profitability index is 1,277.

For proposal B:

$$\begin{aligned} \text{NPV} &= \frac{90\,000}{(1+0{,}12)} + \frac{90\,000}{(1+0{,}12)^2} - 120\,000 \\ &= 32\,105 > 0 \end{aligned}$$

The net present value of the cash flow is R32 105.

$$PI = \frac{152\,105}{120\,000} = 1{,}268 > 1.$$

The profitability index is 1,268.

Thus, on the basis of the net present value, proposal B would be selected (since 32 105 > 27 710); but if the profitability index is used, proposal A must be selected (since 1,277 > 1,268).

Exercise 6.6

(a) Determine the present value of the negative cash flows:

$$PV_{out} = 760 + 80 \times (1 + 0{,}16)^{-1} + 180 \times (1 + 0{,}16)^{-4}$$
$$= 928{,}38$$

The present value of the negative cash flows is R928,38.

(b) Determine future value of the positive cash flows:

$$C = 300 \times (1 + 0{,}19)^4 + 350 \times (1 + 0{,}19)^3 + 600 \times (1 + 0{,}19) + 650$$
$$= 2\,555{,}41$$

The future value of the positive cash flows is R2 555,41.

(c) Determine the MIRR

$$\text{MIRR} = \left(\frac{C}{PV_{out}}\right)^{\frac{1}{n}} - 1$$
$$= \left(\frac{2\,555{,}407}{928{,}378}\right)^{\left(\frac{1}{6}\right)} - 1$$
$$= 18{,}38\%.$$

The MIRR is 18,38%.

Since this is less than 19%, he will probably not decide to go ahead with the project.

Chapter 7. Bonds and debentures

Exercise 7.1

The coupon payment to be received every six months for 18 years is

$$1\,000\,000 \times \frac{0{,}13}{2} = 65\,000.$$

The payment is R65 000.

The stream of coupon payments is
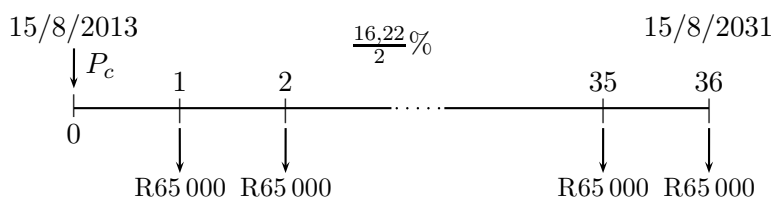
$$P_c = 65\,000a_{\overline{36}|0,1622\div2}$$
$$= 753\,093,48$$

The present value of this stream is R753 093,48.

$$P_f = 1\,000\,000(1 + 0,0811)^{-36}$$
$$= 60\,371,06$$

The present value of the face value is R60 371,06.

$$P = P_c + P_f = 813\,464,54.$$

The present value of the bond is R813 464,54.

Exercise 7.2:1

The time line for the cash flows is as follows:
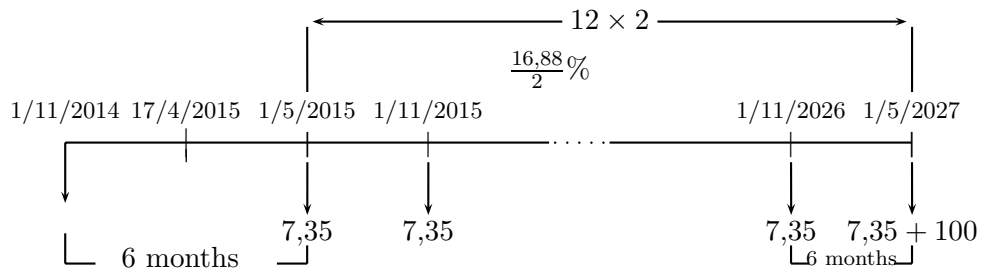
The price on (1/5/2015) is:

$$P = da_{\overline{m}|z} + 100(1 + z)^{-n}$$
$$P(1/5/2015) = 7,35a_{\overline{24}|0,0844} + 100(1 + 0,0844)^{-24}$$
$$= 88,93260$$

The price on the next interest date (1/5/2015) is R88,93260%.

But, since the next interest date is more than ten days after settlement, the bond sells cum and we must include the coupon of 1/5/2015 in the price.

$$88,93260 + 7,35 = 96,28260$$

The total price on 1/5/2015 is R96,28260%.

This must be discounted back to 17 April 2015 as follows:

The remaining number of days from 17/4/2015 to 1/5/2015 is

$$R = 14.$$

The number of days in the half year from 1/11/2014 to 1/5/2015 is $365 - 305$ (1/11) to complete the number of days in 2014 plus 121 (1/5) = 181.

$$H = 181.$$

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The fraction of the half year for discounting is

$$f = \frac{14}{181.}$$

$$
\begin{aligned}
P &= 96{,}28260 \left(1 + \frac{0{,}1688}{2}\right)^{\left(-\frac{14}{181}\right)} \\
&= 95{,}68106
\end{aligned}
$$

The all-in price is R95,68106%.

Exercise 7.2:2

In this case, apart from the fact that the settlement date is seven days later, the only difference from the time line in the previous problem is that the first coupon payment (on 1/5/2015) is missing, since it belongs to the seller – that is, this case is ex interest.

First, as above, we find, that the price on the next interest date is

$$P(1/5/2015) = \text{R88,93260\%}.$$

Since, in this case, the bond sells ex interest, it is not necessary to add in the next coupon payment. All that must be done is to discount this price back to 24 April 2015, that is, for seven days.

$$R = 7 \quad \text{and} \quad H = 181$$

so that

$$f = \frac{7}{181.}$$

$$
\begin{aligned}
P &= 88{,}93260 \left(\frac{1 + 0{,}1688}{2}\right)^{\frac{-7}{181}} \\
&= 88{,}65436
\end{aligned}
$$

The all-in price is R88,65436%.

Note the big difference in price from the previous case.

Exercise 7.3

We first have to determine the clean price for an ordinary coupon of R100 and then multiply the answer by the number of coupons bought, namely $50\,000\,(5\,000\,000 \div 100 = 50\,000)$
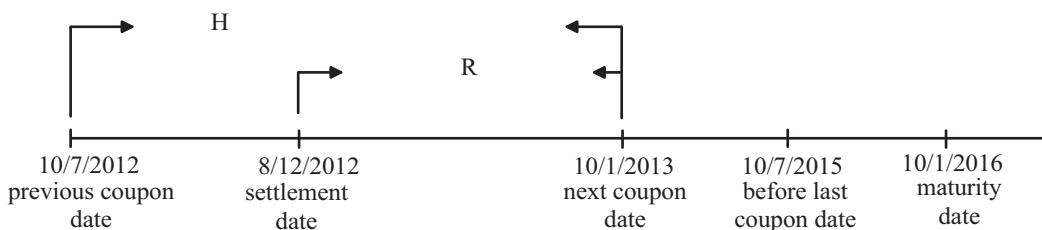
n – the number of coupons still outstanding after the settlement date
- excluding the coupon that follows the settlement date

$$\text{years} = 10/1/2016 - 10/1/2013$$
$$= 6.$$

We must multiply this by two to determine the number of half yearly coupon payments (n)

$$n = 3 \times 2$$
$$= 6.$$

R – the number of days from the settlement day to the next coupon
day, namely day number 365 (31 December) minus day number 342
(8 December) plus day number 10 (10 January) equal 33.

H – the number of days between the previous coupon date until the
next coupon date – the settlement date falls between these two
dates. Day 365 minus day number 191 (10 July) plus day number 10
(10 January) equal 184.

The price on 10/01/2013:

$$P(10/1/2013) = da\,\overline{{}_{n}|}z + 100(1 + z)^{-n}$$
$$= \frac{5}{2}a\,\overline{{}_{6}|}0,12\div 2 + 100\left(1 + \frac{0,12}{2}\right)^{-6}$$
$$= 82{,}78936$$

Because this is a cum interest case, we must add a coupon.

$$P(10/1/2013) = 82{,}78936 + 2{,}5$$
$$= 85{,}28936$$

The price on 10/1/2013 is R85,2896%.

We must now discount this price back to the settlement date.

$$P(8/12/2012) = 85{,}28936\left(1 + \frac{0{,}12}{2}\right)^{\frac{-33}{184}}$$
$$= 84{,}40269$$

The all-in price is R84,40269%.

$$\text{Accrued interest} = \frac{H - R}{365} \times c$$
$$= \frac{184 - 33}{365} \times 5$$
$$= 2{,}06849$$

The accrued interest is R2,06849%.

$$\text{Clean price} = \text{All-in price} - \text{accrued interest}$$
$$= 84{,}40269 - 2{,}06849$$
$$= 82{,}33420$$

The clean price is R82,33420%.

The clean price of a bond with a face value of R5 000 000 equals R4 116 710 (82,33420×
50 000).

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Chapter 8. The handling of data

Exercise 8.1

The arithmetic mean of the *number of days between re-orders* is 28,5 days

$$
\begin{aligned}
\overline{x} &= \frac{\sum_{i=1}^{30} x_i}{30} \\
&= \frac{28 + 36 + 25 + 27 + \ldots + 29 + 32}{30} \\
&= \frac{855}{30} \\
&= 28{,}5
\end{aligned}
$$

Exercise 8.2

The weighted mean is calculated by

$$
\begin{aligned}
\overline{x}_w &= \frac{\sum_{i=1}^{3} x_i w_i}{\sum_{i=1}^{3} w_i} \\
&= \frac{x_1 \times w_1 + x_2 \times w_2 + x_3 \times w_3}{w_1 + w_2 + w_3} \\
&= \frac{0{,}12 \times 1\,000 + 0{,}13 \times 800 + 0{,}15 \times 3\,200}{1\,000 + 800 + 3\,200} \\
&= 0{,}1408 \\
&= 14{,}08\%.
\end{aligned}
$$

Exercise 8.3

The calculations needed to determine the correlation coefficient of the sample are shown.

This calculation can been done directly on your calculator. See Tutorial Letter 101 for the key operations for the prescribed calculators.

Sales-person i	Number of years' experience x_i	Annual sales y_i	x_i^2	$x_i y_i$	y_i^2
1	1	90	1	90	8 100
2	2	107	4	214	11 449
3	3	102	9	306	10 404
4	4	112	16	448	12 544
5	6	113	36	678	12 769
6	8	121	64	968	14 641
7	10	129	100	1 290	16 641
8	10	133	100	1 330	17 689
9	11	134	121	1 474	17 956
10	13	136	169	1 768	18 496
$n = 10$	68	1 177	620	8 566	140 689

From the table it follows that

$$r = \frac{10(8\,566) - 68(1\,177)}{\sqrt{10(620) - (68)^2}\sqrt{10(140\,689) - (1\,177)^2}} = 0,965.$$

The correlation coefficient is close to $+1$, therefore the association between *years' of experience* and *annual sales* is a strong positive correlation. The *number of years' experience* can therefore be used confidently to estimate the *annual sales*.

Exercise 8.4
As calculated in Exercise 8.3, $r = 0,965$. The coefficient of determination is therefore

$$R^2 = 0,965^2 = 0,93$$

This means that 93% of the variation in the annual sales figures can be explained by the number of years' of experience.

APPENDIX B

Solutions to evaluation exercises

> Nothing is a waste of time if you use the experience wisely.
>
> —Auguste Rodin

Chapter 1. Introduction

Chapter 2. Simple interest and discount

1.

$$
\begin{aligned}
S &= P(1 + rt) \\
1 + rt &= \frac{S}{P} \\
rt &= \frac{S}{P} - 1 \\
r &= \frac{\left(\frac{S}{P} - 1\right)}{t}
\end{aligned}
$$

Now $S = 654$, $P = 600$ and $t = \frac{9}{12} = \frac{3}{4}$.

Thus

$$
\begin{aligned}
r &= \frac{\frac{654}{600} - 1}{\frac{3}{4}} \\
&= 0{,}12.
\end{aligned}
$$

The rate of interest is therefore 12% per annum.

2. The future value is obtained using the formula

$$
S = P(1 + rt).
$$

The present value is given by

$$P = \frac{S}{1 + rt}$$

where $S = 1\,500$, $r = 0,16$ and t the time to run to maturity, which, on 1 July, is three months, that is:

$$t = \frac{3}{12} = \frac{1}{4}.$$

Thus

$$
\begin{aligned}
P &= \frac{1\,500}{1 + 0,16 \times \frac{1}{4}} \\
&= 1\,442,31
\end{aligned}
$$

The present value is R1 442,31.

3. We have $S = \text{R}6\,000$, $d = 0,16$ and $t = \frac{9}{12} = \frac{3}{4}$.

   Thus

$$
\begin{aligned}
D &= Sdt \\
&= 6\,000 \times 0,16 \times \frac{3}{4} \\
&= 720
\end{aligned}
$$

   and the discounted value is

$$
\begin{aligned}
P &= S - D \\
&= 6\,000 - 720 \\
&= 5\,280.
\end{aligned}
$$

   The client receives R5 280.

   Now the interest paid is R720.

   Using $I = Prt$ gives $720 = 5\,280 \times r \times \frac{3}{4}$ or

$$
\begin{aligned}
r &= \frac{720}{5\,280} \times \frac{4}{3} \\
&= 0,1818.
\end{aligned}
$$

   OR

$$
\begin{aligned}
r &= \frac{d}{1 - dt} \\
&= \frac{0,16}{1 - 0,16 \times \frac{3}{4}} \\
&= 0,1818.
\end{aligned}
$$

   Thus the equivalent simple interest rate is 18,18% per annum.

4. The following diagram shows the debt, the payments, the dates on which such payments are made, as well as the days to settlement:
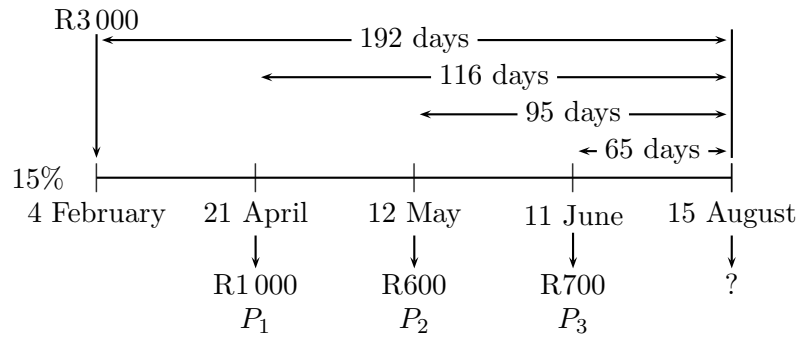
$$\text{Value of R3 000} \;=\; 3\,000 \times \left(1 + 0{,}15 \times \frac{192}{365}\right)$$
$$=\; 3\,236{,}71$$

The value of the debt at the end (15 August) is R3 236,71.

The values of the three payments on 15 August are respectively:

$$P_1 \;=\; 1\,000 \times \left(1 + 0{,}15 \times \frac{116}{365}\right)$$
$$=\; 1\,047{,}67$$

$$P_2 \;=\; 600 \times \left(1 + 0{,}15 \times \frac{95}{365}\right)$$
$$=\; 623{,}42$$

$$P_3 \;=\; 700 \times \left(1 + 0{,}15 \times \frac{65}{365}\right)$$
$$=\; 718{,}70$$

Thus the total value of the payment is

R2 389,79 ($1\,047{,}67 + 623{,}42 + 718{,}70$).

The outstanding debt on 15 August is thus R846,92 ($3\,236{,}71 - 2\,389{,}79$).

Chapter 3. Compound interest

1. For compound interest

$$S = P \left(1 + \frac{j_m}{m}\right)^{tm} \text{ or } S = P\left(1 + i\right)^n$$

If interest is compounded monthly $j_m = 0{,}12$, $m = 12$ and $t = 3$.

$$S \;=\; 3\,000 \left(1 + \frac{0{,}12}{12}\right)^{3 \times 12}$$
$$=\; 4\,292{,}31$$

The accrued principal R4 292,31.

2. We wish to find the present value and hence use

$$P = S \left(1 + \frac{j_m}{m} \right)^{-tm}.$$

Since interest is calculated quarterly $j_m = 0{,}16$, $m = 4$ and $t = 2{,}5$.

$$
\begin{aligned}
P &= 10\,000 \left(1 + \frac{0{,}16}{4} \right)^{-2{,}5 \times 4} \\
&= 6\,755{,}64
\end{aligned}
$$

The amount that must be invested is R6 755,64.

3. We use the formula

$$S = P \left(1 + \frac{j_m}{m} \right)^{tm}$$

with $S = 6\,000$, $P = 4\,000$, $m = 12$ and $t = 3$.

$$
\begin{aligned}
6\,000 &= 4\,000 \left(1 + \frac{j_m}{12} \right)^{(3 \times 12)} \\
1 + \frac{j_m}{12} &= \left(\frac{6\,000}{4\,000} \right)^{\left(\frac{1}{36} \right)} \\
\frac{j_m}{12} &= \left(\frac{6\,000}{4\,000} \right)^{\left(\frac{1}{36} \right)} - 1 \\
j_m &= 12 \left(\left(\frac{6\,000}{4\,000} \right)^{\left(\frac{1}{36} \right)} - 1 \right) \\
&= 0{,}13592
\end{aligned}
$$

The nominal interest rate is thus 13,59% per annum.

4.
$$J_{eff} = 100 \left(\left(1 + \frac{j_m}{m} \right)^m - 1 \right).$$

(a) Half-yearly, with $j_m = 0{,}18$ and $m = 2$.

$$
\begin{aligned}
J_{eff} &= 100 \left(\left(1 + \frac{0{,}18}{2} \right)^2 - 1 \right) \\
&= 18{,}81\%
\end{aligned}
$$

The effective interest rate is 18,81% per annum.

(b) Monthly, with $j_m = 0{,}18$ and $m = 12$.

$$
\begin{aligned}
J_{eff} &= 100 \left(\left(1 + \frac{0{,}18}{12} \right)^{12} - 1 \right) \\
&= 19{,}56\%
\end{aligned}
$$

The effective interest rate is 19,56% per annum.

5. The timeline is:
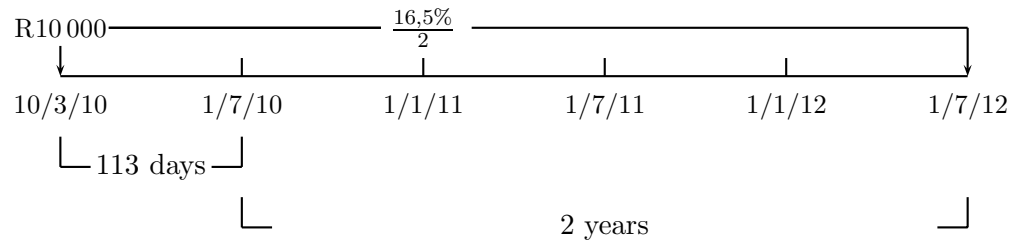
First, we need to calculate the number of days from 10 March 2010 to 1 July 2010 (the first compounding date). We find (check!):

Number of days: 113

(a) If simple interest followed by compound interest is used:

$$
\begin{aligned}
S &= P(1 + rt)\left(1 + \frac{j_m}{m}\right)^{tm} \\
&= 10\,000 \times \left(1 + 0{,}165 \times \frac{113}{365}\right) \times \left(1 + \frac{0{,}165}{2}\right)^{2 \times 2} \\
&= 14\,432{,}72
\end{aligned}
$$

The accumulated amount is R14 432,72.

Note that there are four compounding periods in the term from 1 July 2010 to 1 July 2012, that is $t = 2$ and $m = 2$.

(b) We use the formula

$$
S = P\left(1 + \frac{j_m}{m}\right)^{tm}
$$

with $j_m = 0{,}165$, $m = 2$ and $t = 2 + \frac{113}{365}$.

$$
\begin{aligned}
S &= 10\,000\left(1 + \frac{0{,}165}{2}\right)^{\left(\left(2 + \frac{113}{365}\right) \times 2\right)} \\
&= 14\,422{,}10
\end{aligned}
$$

The accumulated amount is R14 422,10.

6. (a) We use the formula

$$
i = n\left(\left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1\right)
$$

with $n = 4$, $j_m = 12\%$ and $m = 12$

$$
\begin{aligned}
i &= 4\left(\left(1 + \frac{0{,}12}{12}\right)^{\frac{12}{4}} - 1\right) \\
&= 0{,}121204
\end{aligned}
$$

The converted interest rate is 12,1204%

(b) The second problem is solved in the same way:

$$i = n\left(\left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1\right)$$

with $m = 2$, $n = 365$ and $j_m = 18\%$

$$
\begin{aligned}
&= 365\left(\left(1 + \frac{0{,}18}{2}\right)^{\frac{2}{365}} - 1\right)\\
&= 0{,}172396\\
&= 17{,}2396\%
\end{aligned}
$$

The converted interest rate is 17,2396%

7. Since the maturity values are given, we can immediately draw the time line.
   Note that if we denote his first payment one year from now by X, then the
   second one, five years later (ie six years hence), is $2X$, since Paul wishes to
   make it twice as much as the first.

   The time line is as follows:
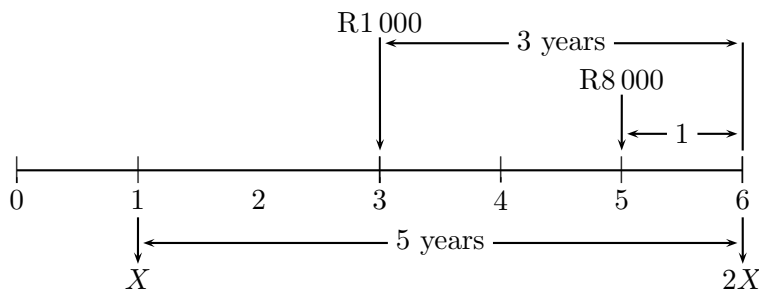
   Take the new due date (year six) as the comparison date with $j_m = 0{,}18$, $m = 4$
   and $t =$ number of years' for each obligation or payment.

$$X\left(1 + \frac{0{,}18}{4}\right)^{5\times 4} + 2X = 1\,000\left(1 + \frac{0{,}18}{4}\right)^{3\times 4} + 8\,000\left(1 + \frac{0{,}18}{4}\right)^{1\times 4}.$$

   Take X out as the common factor:

$$X\left(\left(1 + \frac{0{,}18}{4}\right)^{5\times 4} + 2\right) = 1\,000\left(1 + \frac{0{,}18}{4}\right)^{3\times 4} + 8\,000\left(1 + \frac{0{,}18}{4}\right)^{1\times 4}$$

$$
\begin{aligned}
X &= \frac{\left(1\,000\left(1 + \frac{0{,}18}{4}\right)^{3\times 4} + 8\,000\left(1 + \frac{0{,}18}{4}\right)^{1\times 4}\right)}{\left(\left(1 + \frac{0{,}18}{4}\right)^{5\times 4} + 2\right)}\\
&= 2\,546{,}86
\end{aligned}
$$

Thus the first payment in one year is R2 546,86, while the one five years later is double this, namely R5 093,72.

Check:

$$2\,546{,}86\left(1+\frac{0{,}18}{4}\right)^{20}+5\,093{,}72=11\,236{,}03.$$

and

$$1\,000\left(1+\frac{0{,}18}{4}\right)^{3\times4}+8\,000\left(1+\frac{0{,}18}{4}\right)^{1\times4}=11\,236{,}03$$

Chapter 4. Annuities

1. The time line is:
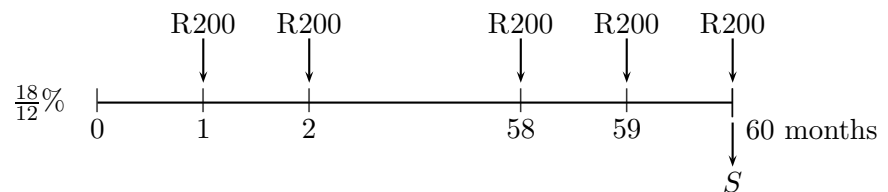
The interest rate per month is $i=0{,}18\div12$, $R=200$ and the term is $n=5\times12$.

$$\begin{aligned}S&=200s_{\overline{5\times12}|0{,}18\div12}\\&=19\,242{,}93\end{aligned}$$

The future amount is R19 242,93.

$$\begin{aligned}P&=200a_{\overline{5\times12}|0{,}18\div12}\\&=7\,876{,}05\end{aligned}$$

The present value is R7 876,05.

The interest received is the difference between the future value and the total amount actually paid.

$$\begin{aligned}I&=19\,242{,}93-60\times200\\&=7\,242{,}93\end{aligned}$$

The interest received is R7 242,93.

2. The time line is now
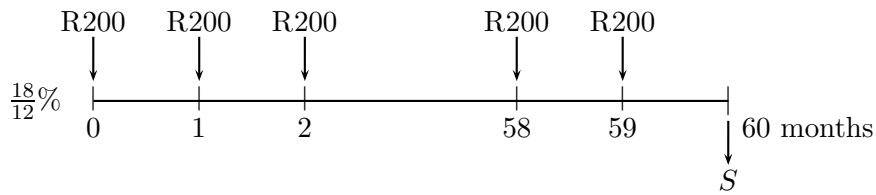
$$S = (1+i)Rs_{\overline{n}|i}$$
$$= (1+0,015) \times 200s_{\overline{5 \times 12}|0,18 \div 12}$$
$$= 19\,531,57$$

The future value is R19 531,57.

$$P = S(1 + \frac{j_m}{m})^{tm}$$
$$= 19531,57 \left(1 + \frac{0,18}{12}\right)^{-5 \times 12}$$
$$= 19531,57(1 + 0,015)^{-60}$$
$$= 7\,994,19$$

The present value is R7 994,19.

$$I = 60 \times 200 - 7\,994,19$$
$$= 4\,005,81$$

The interest paid is R4 005,81.

3. The time line is:
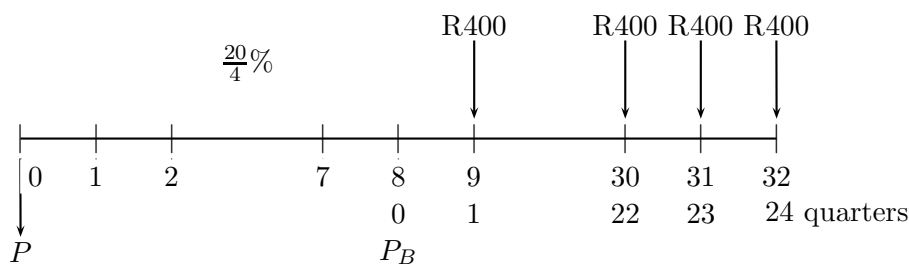
To obtain P, we must first determine the present value of the annuity at year two and then discount it back to now.

$$P = Ra_{\overline{n}|i}(1 + \frac{j_m}{m})^{-tm}$$
$$= 400a_{\overline{6 \times 4}|0,20 \div 4} \left(1 + \frac{0,20}{4}\right)^{-2 \times 4}$$
$$= 3\,735,79$$

The present value at time zero is R3 735,79.

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4. The time line for the problem is
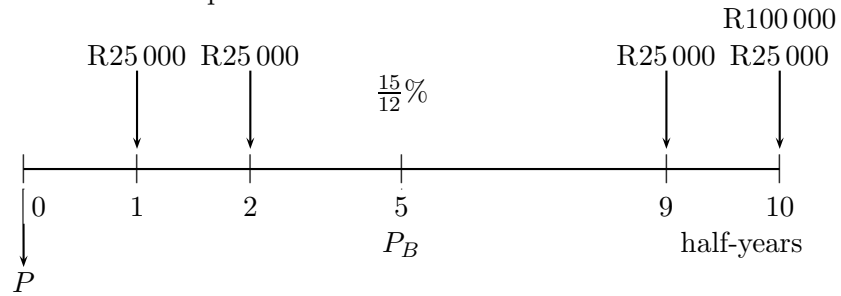
If we ignore the final R100 000 payment, then, except for the interest compounding, the structure is that of an ordinary annuity. We must convert the 15% nominal interest rate, compounded monthly, to the half-yearly equivalent.

$$
\begin{aligned}
j_n &= n\left(\left(1+\frac{j_m}{m}\right)^{\frac{m}{n}}-1\right) \\
&= 2\left(\left(1+\frac{0,15}{12}\right)^{\frac{12}{2}}-1\right) \\
&= 0,15476 \\
&= 15,4766\ldots\%.
\end{aligned}
$$

But this is the nominal annual rate for half-yearly compounding.

The payments of R25 000 now constitute an ordinary annuity at an interest rate of 15,4766 ...% per annum, compounded semi-annually.

$$
\begin{aligned}
P_a &= 25\,000a_{\overline{5\times2}|0,154766\cdots\div2} \\
&= 169\,750,19
\end{aligned}
$$

The present value is R169 750,19.

There is, however, an additional final payment of R100 000 at the end. Its present value is simply R100 000 discounted back for five years at 15,4766% per annum, compounded semi-annually

$$
\begin{aligned}
P_f &= 100\,000(1+\frac{0,154766\ldots}{2})^{-5\times2} \\
&= 47\,456,76
\end{aligned}
$$

(You would get the same result if you discounted the R100 000 back for 60 periods at 1,25% $(0,15 \div 12)$ per period.)

The total present value of the contract is the sum of P_a and P_f: The total present value is R217 206,95 (169 750,19 + 47 456,76).

5.

$$
\begin{aligned}
\text{The first payment for the annuity} &= R \\
\text{The second payment for the annuity} &= (1+r)R \\
\text{The third payment for the annuity} &= (1+r)(1+r)R \\
&= (1+r)^2R \\
&\;\;\vdots \\
\text{The last } (n\text{th}) \text{ payment for the annuity} &= (1+r)^{n-1}R.
\end{aligned}
$$

The time line is:
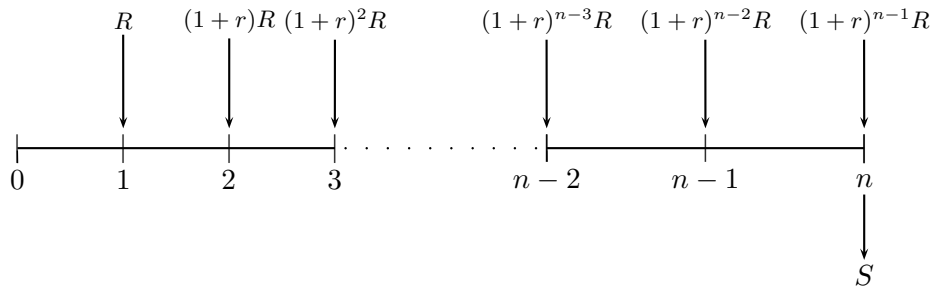
Now

$$
\begin{aligned}
S \;=\; & R(1+i)^{n-1} \\
& +(1+r)R(1+i)^{n-2} \\
& +(1+r)^2 R(1+i)^{n-3} \\
& \;\;\vdots \\
& +(1+r)^{n-2}R(1+i) \\
& +(1+r)^{n-1}R.
\end{aligned}
$$

Thus

$$
\begin{aligned}
S \;=\; & R(1+i)^{n-1} \\
& +R(1+i)^{n-2}(1+r) \\
& +R(1+i)^{n-3}(1+r)^2 \\
& \;\;\vdots \\
& +R(1+i)(1+r)^{n-2} \\
& +R(1+r)^{n-1}. \qquad\qquad \text{(B.1)}
\end{aligned}
$$

Now multiply both sides of $(B.1)$ by $(1+i)(1+r)^{-1}$. This gives:

$$
\begin{aligned}
\frac{1+i}{1+r}S \;=\; & R(1+i)^{n}(1+r)^{-1} \\
& +R(1+i)^{n-1} \\
& +R(1+i)^{n-2}(1+r) \\
& \;\;\vdots \\
& +R(1+i)^2(1+r)^{n-3} \\
& +R(1+i)(1+r)^{n-2}. \qquad\qquad \text{(B.2)}
\end{aligned}
$$

After subtracting $(B.1)$ from $(B.2)$ the left- hand side of the equation will equal:

$$
\begin{aligned}
\frac{1+i}{1+r}S - S \;=\; & \frac{1+i}{1+r}S - \frac{1+r}{1+r}S \\
=\; & \frac{1}{1+r}S(1+i-(1+r)) \\
=\; & \frac{1}{1+r}S(i-r)
\end{aligned}
$$

and the right-hand side will equal:

$$R(1+i)^n(1+r)^{-1} - R(1+r)^{n-1}$$
$$= R(1+i)^n(1+r)^{-1} - R(1+r)^n(1+r)^{-1}$$
$$= (1+r)^{-1}R\left((1+i)^n - (1+r)^n\right) \qquad \text{(Take out the common factor)}$$
$$= \frac{R\left((1+i)^n - (1+r)^n\right)}{(1+r)}.$$

Thus

$$\frac{1}{1+r}S(i-r) = \frac{R\left((1+i)^n - (1+r)^n\right)}{(1+r)}$$
$$S = \frac{R\left((1+i)^n - (1+r)^n\right)}{i-r}.$$

Chapter 5. Amortisation and sinking funds

1. The principal borrowed is R125 000 (475 000 − 350 000). The number of payments is 12×20 and the interest rate is $19\% \div 12$.

$$R = \frac{125\,000}{a_{\overline{20 \times 12}|0,19 \div 12}}$$
$$= 2\,025{,}86$$

The monthly payments are R2 2025,86.

$$P = 2\,025{,}86\,a_{\overline{(20-5) \times 12}|0,19 \div 12}$$
$$= 120\,380{,}59$$

After five years, the present value will be R120 380,59.

Thus, they have paid off R4 619,41(125 000 − 120 380,59). Adding to this their down payment of R350 000, we find that the equity after five years is R354 619,41.

At this stage, the interest rate is adjusted to 18% per annum compounding monthly, and the outstanding principal of R120 380,59 must be repaid over the remaining 15 years at the new interest rate.

$$R = \frac{120\,380{,}59}{a_{\overline{(20-5) \times 12}|0,18 \div 12}}$$
$$= 1\,938{,}63$$

The adjusted payments are R1 938,63.

2. The annual payments on a loan of R12 000 for five years at 10% interest compounded annually are:

$$R = \frac{P}{a_{\overline{n}|i}}$$

$$R = \frac{12\,000}{a_{\overline{5}|0,1}}$$

$$= 3\,165{,}57$$

The annual payment is R3 165,57.

The amortisation schedule is as follows:

Year	Outstanding principal at year beginning	Interest due at year end	Payment	Principal repaid	Principal at year end
1	12 000,00	1 200,00	3 165,57	1 965,57	10 034,43
2	10 034,43	1 003,44	3 165,57	2 162,13	7 872,30
3	7 872,30	787,23	3 165,57	2 378,34	5 493,96
4	5 493,96	549,40	3 165,57	2 616,17	2 877,79
5	2 877,79	287,78	3 165,57	2 877,79	0,00
Total		3 827,85	15 827,85	12 000,00	

3. The principal that you will have to borrow is R680 000 (960 000 − 280 000), at 20% per annum, compounded monthly over 25 years,

$$R = \frac{680\,000}{a_{\overline{25\times12}|0,20\div12}}$$

$$= 11\,413{,}47$$

The monthly payments are R11 413,47.

This is the total monthly payment. After deducting the R2 000 per month subsidy, the monthly payment is R9 413,47.

The total interest paid over the term of the loan is:

$$I = \text{number of payments} \times \text{payment} - \text{the loan}$$

$$= 300 \times 11\,413{,}47 - 680\,000$$

$$= 3\,424\,041 - 680\,000$$

$$= 2\,744\,041$$

The total interest that must be paid over the term of the loan of 25 years is R2 744 041,00.

Assuming an inflation rate of 0,75% per month – that is 9%(0,75% × 12) per year, compounded monthly.

The total value of the loan:

$$T_\nu = Ra_{\overline{n}|r}$$

$$= 11\,413{,}47 a_{\overline{25\times12}|0,09\div12}$$

$$= 1\,360\,047{,}84$$

The total real cost of the loan including the subsidy is R1 360 047,84.

$$
\begin{aligned}
T_r &= T_\nu - P \\
&= 1\,360\,047,84 - 680\,000 \\
&= 680\,047,84
\end{aligned}
$$

Thus the additional amount paid, over and above the principal, in terms of the rand of today is R680 047,84 which is only about a quarter of the interest paid.

$$
\begin{aligned}
T_\nu &= 9\,413,47 a_{\overline{25 \times 12}|0,09 \div 12} \\
&= 1\,121\,724,36
\end{aligned}
$$

The total value of the loan considering your part of the payment is R1 121 724,36.

The total real cost of the loan excluding the subsidy is:

$$
\begin{aligned}
T_r &= T_\nu - P \\
&= 1\,121\,724,36 - 680\,000 \\
&= 441\,724,36
\end{aligned}
$$

This is still R441 724,36 more than the principal borrowed but, if you have chosen wisely, your house should appreciate faster than the expected average inflation rate. On the other hand, if inflation is higher than 0,75% per month, the proposition will look even better.

4. In respect of the first plan, the interest rate is 15% per annum and the term five years, whereby the loan is to be redeemed by amortisation, are

$$
\begin{aligned}
R &= \frac{50\,000}{a_{\overline{5}|0,15}} \\
&= 14\,915,78
\end{aligned}
$$

The yearly payments are R14 915,78.

In respect of the sinking fund plan, the interest rate is of 14% of the principal, plus deposits based on an 11% interest rate.

$$
\begin{aligned}
R &= 50\,000 \times 0,14 + \left(\frac{50\,000}{s_{\overline{5}|0,11}} \right) \\
&= 7\,000 + 8\,028,52 \\
&= 15\,028,52
\end{aligned}
$$

The yearly payments are R15 028,52.

Thus, the sinking fund plan will cost Mr Deal R112,74 (15 028,52 − 14 915,78) per year more than the first plan and he should therefore approach the Now Bank for Tomorrow for his loan.

Chapter 6. Evaluation of Cash flows

1. (a) Internal rate of return:
   Project A:

   $$f(i) = \frac{300}{1+i} + \frac{330}{(1+i)^2} + \frac{330}{(1+i)^3} + \frac{300}{(1+i)^4} - 760$$

   Solving numerically gives $i = 24\%$.
   Project B:

   $$\begin{aligned} f(i) &= 310a_{\overline{4}|i} - 800 \\ &= \frac{310}{1+i} + \frac{310}{(1+i)^2} + \frac{310}{(1+i)^3} + \frac{310}{(1+i)^4} - 800 \end{aligned}$$

   Solving yields $i = 20\%$.
   Project C:

   $$f(i) = \frac{400}{1+i} + \frac{420}{(1+i)^2} + \frac{440}{(1+i)^3} - 800$$

   Solving numerically gives $i = 26\%$.
   Thus, using the internal rate of return as the basis, proposal B will be rejected, since $i < K$ (ie $20 < 21$), while A and C are acceptable.

   (b) Net present value:
   Project A:

   $$\begin{aligned} NPV &= \frac{300}{1+0{,}21} + \frac{330}{(1+0{,}21)^2} + \frac{330}{(1+0{,}21)^3} + \frac{300}{(1+0{,}21)^4} - 760 \\ &= 40 \end{aligned}$$

   The NPV is R40 000.
   Project B:

   $$P = 310a_{\overline{4}|0{,}21} = 788$$

   and
   $$NPV = 788 - 800 = -12.$$

   The NPV is $-$R12 000.
   Project C:

   $$\begin{aligned} NPV &= \frac{400}{1+0{,}21} + \frac{420}{(1+0{,}21)^2} + \frac{440}{(1+0{,}21)^3} - 800 \\ &= 66 \end{aligned}$$

   The NPV is R66 000.
   With the net present value as the basis, B will be rejected. Furthermore, we would advise that C be selected instead of A since it has the greatest NPV.

(c) Profitability index:

We already have the net present values so we can calculate the profitability index without further ado:

Project A:

$$\begin{aligned} PI & = \frac{NPV + \text{ outlays}}{\text{outlays}} \\ & = \frac{40 + 760}{760} \\ & = 1{,}053 \end{aligned}$$

The profitability index is 1,053.

Project B:

$$\begin{aligned} PI & = \frac{-12 + 800}{800} \\ & = 0{,}985 \end{aligned}$$

The profitability index is 0,985.

Project C:

$$\begin{aligned} PI & = \frac{66 + 800}{800} \\ & = 1{,}083 \end{aligned}$$

The profitability index is 1,083.

Our advice is once again the same. Note that $PI < 1$ for case B, which means that we should reject it.

2. Project X:

$$PV_{out} = 800 + 40(1 + 0{,}22)^{-1} = 832{,}79$$

The present value of the investments is R832,79.

$$\begin{aligned} C & = 200 \times (1 + 0{,}19)^3 + 510 \times (1 + 0{,}19)^2 + 510 \times (1 + 0{,}19)^1 + 510 \\ & = 2\,176{,}14 \end{aligned}$$

The FV of the returns is R2 176,14.

$$\begin{aligned} MIRR & = \left(\frac{C}{PV_{out}}\right)^{\frac{1}{n}} - 1 \\ & = \left(\frac{2\,176\,14}{832{,}79}\right)^{\left(\frac{1}{5}\right)} - 1 \\ & = 21{,}18\% \end{aligned}$$

The MIRR is 21,18%.

Project Y:

$$PV_{out} = 600 + 200(1 + 0{,}22)^{-1} = 763{,}93$$

The present value is R763,93.

$$
\begin{aligned}
C &= 440 \times (1 + 0{,}19)^2 + 440 \times (1 + 0{,}19) + 440 \\
&= 1\,586{,}68
\end{aligned}
$$

The future value is R1 586,68.

$$
\begin{aligned}
MIRR &= \left(\frac{1\,586{,}68}{763{,}93} \right)^{\left(\frac{1}{4}\right)} - 1 \\
&= 20{,}05\%
\end{aligned}
$$

The MIRR is 20,05%.

Thus, using the MIRR criterion, project X should be selected.

Chapter 7. Bonds and debentures

1. (a) Settlement date: 25 September 2012.

    The time line for the cash flows is as follows:
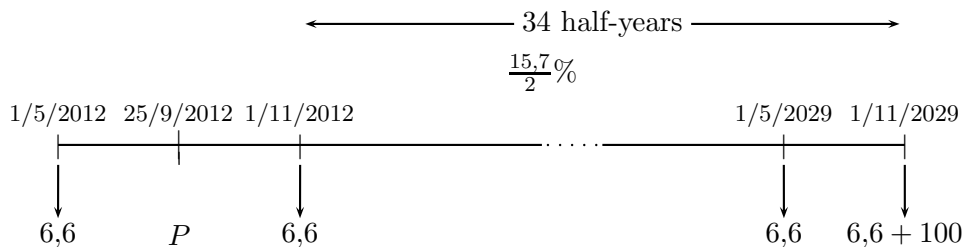
    The price on the next coupon date is:

$$
\begin{aligned}
P &= da_{\overline{n}|z} + 100(1 + z)^{-n} \\
P(1/11/2012) &= 6{,}6 \times a_{\overline{34}|0{,}0785} + 100(1 + 0{,}0785)^{-34} \\
&= 85{,}29586
\end{aligned}
$$

    Since this is cum interest, add the coupon of R6,6% to get R91,89586%.

    The remaining number of days from 25/9/2012 to 1/11/2012 is

$$
R = 37.
$$

    The number of days in the half year from 1/5/2012 to 1/11/2012 is

$$
H = 184.
$$

    The fraction of the half year for discounting is

$$
f = \frac{37}{184}.
$$

$$P = 91{,}89586 \times (1 + 0{,}0785)^{(-37/184)}$$
$$= 90{,}50993$$

The all-in price is R90,50993%.

$$
\begin{aligned}
\text{Accrued interest} &= \frac{(H - R)}{365} \times c \\
&= \frac{184 - 37}{365} \times 13{,}2 \\
&= 5{,}31616
\end{aligned}
$$

The accrued interest is R5,31616%.

$$
\begin{aligned}
\text{Clean price} &= \text{All-in price} - \text{Accrued interest} \\
&= 90{,}50993 - 5{,}31616 \\
&= 85{,}19377
\end{aligned}
$$

The clean price is R85,19377%.

(b) Settlement date: 25 October 2012.

This is similar to the above problem, except that the settlement date is just over a week later and within ten days of the next coupon. Hence the price is ex interest.

Again, we start with the price on the next coupon date:

$$P(1/11/2012) = \text{R}85{,}29586\%$$

but, in this case, no coupon is added.

The remaining number of days from 25/10/2012 to 1/11/2012 is

$$R = 7.$$

The number of days in the half year (see above) is

$$H = 184.$$

The fraction of the half year for discounting is

$$f = \frac{7}{184}.$$

$$P = 85{,}29586 \times (1 + 0{,}0785)^{(-7/184)}$$
$$= 85{,}05099$$

The all-in price is R85,05099%.

$$\begin{aligned}
\text{Accrued interest} \quad &= \quad \frac{-R}{365} \times c \\
&= \quad \frac{-7}{365} \times 13{,}2 \\
&= \quad -0{,}25315
\end{aligned}$$

The accrued interest is now $-$R0,25315%.

$$\begin{aligned}
\text{Clean price} \quad &= \quad \text{All-in price} - \text{Accrued interest} \\
&= \quad 85{,}05099 - (-0{,}25315) \\
&= \quad 85{,}30414
\end{aligned}$$

The clean price is R85,30414%.

Chapter 8. The handling of data

1. To find the average price this motorist paid per litre of petrol, use the weighted mean formula, putting $x_1 = 10{,}14, x_2 = 10{,}03, x_3 = 10{,}20$ and $w_1 = 12, w_2 = 18, w_3 = 50$. Here the weights w_1, w_2, and w_3 express the relative importance of the three prices of petrol, and upon substitution we find:

$$\begin{aligned}
\overline{x}_w \quad &= \quad \frac{\sum_{i=1}^{3} x_i w_i}{\sum_{i=1}^{3} w_i} \\
&= \quad \frac{10{,}14 \times 12 + 10{,}03 \times 18 + 10{,}20 \times 50}{12 + 18 + 50} \\
&= \quad \frac{812{,}22}{80} \\
&= \quad 10{,}15.
\end{aligned}$$

The average petrol price is R10,15 per litre.

2. The mean percentage growth in rand has to be calculated before the standard deviation can be computed.

   (a) The mean for Bank A is:

$$\begin{aligned}
\overline{x}_A \quad &= \quad \frac{\sum_{i=1}^{4} x_i}{4} \\
&= \quad \frac{10{,}9 + 11{,}7 + 24{,}3 + 0{,}2}{4} \\
&= \quad 11{,}775
\end{aligned}$$

and the standard deviation is:

$$
\begin{aligned}
S_A &= \sqrt{\frac{\sum_{i=1}^{4}(x_i - \overline{x})^2}{4-1}} \\
&= \sqrt{\frac{(10,9 - 11,775)^2 + (11,7 - 11,775)^2 + (24,3 - 11,775)^2 + (0,2 - 11,775)^2}{3}} \\
&= \sqrt{\frac{291,6275}{3}} \\
&= 9,86
\end{aligned}
$$

For Bank B:
The mean:

$$
\begin{aligned}
\overline{x}_B &= \frac{56,3}{4} \\
&= 14,075
\end{aligned}
$$

The standard deviation:

$$
\begin{aligned}
S_B &= \sqrt{\frac{273,7475}{3}} \\
&= 9,55
\end{aligned}
$$

For Bank C:
The mean:

$$
\begin{aligned}
\overline{x}_C &= \frac{35,6}{4} \\
&= 8,9
\end{aligned}
$$

The standard deviation:

$$
\begin{aligned}
S_C &= \sqrt{\frac{105,4200}{3}} \\
&= 5,93
\end{aligned}
$$

   (b) Bank A has the greatest variation in percentage growth because it has the highest standard deviation, namely 9,86. Bank B performs better than Bank A because it has a higher mean percentage growth (14,075 compared to 11,775) and the percentage growth is more stable (standard deviation is 9,55 which is lower than that of Bank A). Overall, Bank C has the lowest variation in percentage growth with a standard deviation of 5,93, but its mean percentage growth is also much lower than the other two banks, namely 8,9%.

3. (a) Substitute $x = 5$ into $y = 10\,210 + 850x$. This gives

$$
\begin{aligned}
y &= 10\,210 + 850 \times 5 \\
&= 14\,460.
\end{aligned}
$$

The estimated monthly income of an employee with five years of service is R14 460.

(b) The slope of the function is 850. Thus for each year of additional service an employee can expect a raise of R850.

4. (a)

It looks like there might be a linear relationship between the two variables.

(b) $y = a + bx$ where

$$b = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a = \frac{\sum y_i}{n} - \frac{b \sum x_i}{n}$$

From the previous table we find that

$$b = \frac{12(83\,627{,}25) - (224{,}75)(4\,520)}{12(4\,239{,}938) - (224{,}75)^2}$$

$$b = -33{,}66026$$

and

$$a = \frac{4\,520}{12} - \frac{-33{,}6603(224{,}75)}{12}$$

$$a = 1\,007{,}096035.$$

Thus $y = 1\,007{,}096 - 33{,}66x$.

(c) If the mortgage rate is 18%, the number of contract is equal to

$$y = 1\,007{,}096 - 33{,}66(18) = 401{,}216.$$

If the mortgage rate is 16,5%, the number of contract is equal to

$$y = 1\,007{,}096 - 33{,}66(16{,}5) = 451{,}706.$$

Thus the number of contracts will increase by $451{,}706 - 401{,}216 = 50{,}49 \approx 51$.

Note: You could also have used your calculator to calculate question b and c. It is much quicker.

(d)

n	x_j	y_i	x_i^2	y_i^2	$x_i x y_i$
1	16,00	443	256,0000	196 249	7 088,00
2	16,75	448	280,5625	200 704	7 504,00
3	17,00	367	289,0000	134 689	6 239,00
4	17,50	492	306,2500	242 064	8 610,00
5	18,25	384	333,0625	147 456	7 008,00
6	18,75	437	351,5625	190 969	8 193,75
7	19,00	356	361,0000	126 736	6 764,00
8	19,50	339	380,2500	114 921	6 610,50
9	19,75	365	390,0625	133 225	7 208,75
10	20,25	321	410,0625	103 041	6 500,25
11	20,75	338	430,5625	114 244	7 013,50
12	21,25	230	451,5625	52 900	4 887,50
Total	224,75	4 520	4 239,9380	1 757 198	83 627,25

$$r = \frac{n\sum_{i=1}^{n} - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} y_i)^2}}$$

$$r = \frac{12(83\,627{,}25) - (224{,}75)(4\,520)}{\sqrt{12(4\,239{,}938) - (224{,}75)^2}\sqrt{12(1\,757\,198) - (4\,520)^2}}$$

$$= \frac{-12\,343}{15\,509{,}42086}$$

$$r = -0{,}795838871$$

$$r \approx -0{,}80$$

There exists a negative linear relationship between the mortgage rate and the number of building contracts undertaken.

The number of each day of the year

	FOR LEAP YEARS ADD ONE TO THE NUMBER EVERY DAY AFTER FEBRUARY 28												
Day	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**	**Day**
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29		88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

Formulæ

Simple interest

$$S = P(1 + rt)$$

$S \equiv$ accumulated amount / future value
$P \equiv$ principal / present value
$r \equiv$ interest rate per year
$t \equiv$ time in years

Simple discount

$$P = S(1 - dt)$$

$P \equiv$ principal / present value
$S \equiv$ accumulated amount / future value
$d \equiv$ discount rate per year
$t \equiv$ time in years

Compound interest

$$S = P(1 + \frac{j_m}{m})^{tm}$$

$S \equiv$ accumulated amount / future value
$P \equiv$ principal / present value
$j_m \equiv$ nominal interest rate per year
$m \equiv$ number of compounding periods per year
$t \equiv$ time in years

OR

$$S = P(1 + i)^n$$

$S \equiv$ accumulated amount / future value
$P \equiv$ principal / present value
$i \equiv$ nominal interest rate per compounding period
$n \equiv$ number of compounding periods

Effective interest rate

$$J_{eff} = 100 \left(\left(1 + \frac{j_m}{m} \right)^m - 1 \right)$$

$J_{eff} \equiv$ effective interest rate (as a percentage)
$j_m \equiv$ nominal interest rate per year
$m \equiv$ number of compounding periods per year

Continuous compounding

$$c = m \ln\left(1 + \frac{j_m}{m}\right)$$

c	$\equiv$	continuous compounding rate
j_m	$\equiv$	nominal interest rate per year
m	$\equiv$	number of compounding periods per year
ln	$\equiv$	ln function on the calculator

$$S = Pe^{ct}$$

S	$\equiv$	accumulated amount / future value
P	$\equiv$	principal / present value
e	$\equiv$	e^x exponential function (on calculator)
c	$\equiv$	continuous compounding rate
t	$\equiv$	time in years

Converting interest

$$i = n\left(\left(1 + \frac{j_m}{m}\right)^{\frac{m}{n}} - 1\right)$$

i	$\equiv$	converted interest rate per period
n	$\equiv$	number of compounding periods
j_m	$\equiv$	given nominal interest rate per year
m	$\equiv$	number of compounding periods per year of the given nominal rate

Converting continuous compounding to effective interest rate

$$J_\infty = 100(e^c - 1)$$

J_∞	$\equiv$	effective interest rate for continuous compounding (as a percentage)
e	$\equiv$	e^x exponential function (on calculator)
c	$\equiv$	continuous compounding rate

Future value of an annuity

$$S = R\left[\frac{(1+i)^n - 1}{i}\right]$$

S	$\equiv$	accumulated amount / future value
R	$\equiv$	equal payment (amounts) paid at equal intervals
i	$\equiv$	interest rate per year divided by the number of compounding periods per year
n	$\equiv$	number of years multiplied by the number of compounding periods per year

Present value of an annuity

$$P = R\left(\frac{(1+i)^n - 1}{i(1+i)^n}\right)$$

P	$\equiv$	principal / present value
R	$\equiv$	equal (payment) amounts paid at equal intervals
i	$\equiv$	interest rate per year divided by the number of compounding periods per year
n	$\equiv$	number of years multiplied by the number of compounding periods per year

Annuity due

$$P = (1+i)Ra_{\overline{n}|i} \quad \text{or} \quad S = (1+i)Rs_{\overline{n}|i}$$

Multiply the answer of the present value or future value of an annuity with $(1+i)$.

Increasing annuity

$$S = \left(R + \frac{Q}{i} \right) s_{\overline{n}|i} - \frac{nQ}{i}$$

S	$\equiv$	accumulated amount / future value
R	$\equiv$	equal (payment) amounts paid at equal intervals
Q	$\equiv$	amount by which the regular payments will increase every period
i	$\equiv$	interest rate per period
n	$\equiv$	number of compounding periods
$s_{\overline{n}|i}$	$\equiv$	$\frac{(1+i)^n - 1}{i}$; an ordinary annuity

Perpetuity

$$P = \frac{R}{i}$$

P	$\equiv$	present value / principal
R	$\equiv$	equal payment per period
i	$\equiv$	interest rate per period

The relationship between the future value and the present value of an annuity:

$$s_{\overline{n}|i} \equiv (1+i)^n a_{\overline{n}|i}$$

$s_{\overline{n}|i}$ is the future value of $a_{\overline{n}|i}$ and $a_{\overline{n}|i}$ is the present value of $s_{\overline{n}|i}$.

The real cost of a loan

$$T_r = Ra_{\overline{n}|r} - P$$

T_r	$\equiv$	total real cost of the loan
R	$\equiv$	equal amount (payment) paid at equal intervals
n	$\equiv$	total number of compounding periods
r	$\equiv$	inflation rate per period
P	$\equiv$	original loan

Internal rate of return

$$\frac{C_1}{(1+i)^1} + \frac{C_2}{(1+i)^2} + \frac{C_3}{(1+i)^3} + \ldots + \frac{C_m}{(1+i)^m} - I_{out} = 0$$

$C_1, \ldots C_m$	$\equiv$	cash inflows in the sequence in which they appear
i	$\equiv$	internal rate of return
I_{out}	$\equiv$	initial investment

Net present value

$$\text{NPV} = \frac{C_1}{(1+K)^1} + \frac{C_2}{(1+K)^2} + \frac{C_3}{(1+K)^3} + \ldots + \frac{C_m}{(1+K)^m} - I_{out}$$

NPV	$\equiv$	net present value
$C_1, \ldots C_m$	$\equiv$	cash flows in the sequence in which they appear
K	$\equiv$	cost of capital, that is, the interest rate at which money can be borrowed
I_{out}	$\equiv$	initial investment

Profitability index

$$\text{PI} = \frac{\text{Present value of cash inflows}}{\text{Present value of cash outflows}}$$

OR

$$\text{PI} = \frac{\text{NPV + outlays (initial investment)}}{\text{outlays (initial investment)}}$$

Modified internal rate of return

$$\text{MIRR} = \left(\frac{C}{PV_{out}}\right)^{\frac{1}{n}} - 1$$

MIRR	$\equiv$	modified internal rate of return
C	$\equiv$	future value of all the positive cashflows
PV_{out}	$\equiv$	present value of all the negative cash flows
n	$\equiv$	the lifetime of the project in time periods

Bonds

$$P = da\,\overline{{}_{n}|_z} + 100(1+z)^{-n}$$

P	$\equiv$	present value of the stock
d	$\equiv$	half-yearly coupon rate
z	$\equiv$	half-yearly yield to maturity
n	$\equiv$	number of coupons still outstanding after the settlement - excluding the coupon date that follows the settlement date

Accrued interest

Cum interest	$=$	$\frac{H-R}{365} \times c$
ex interest	$=$	$\frac{-R}{365} \times c$
H	$\equiv$	number of days between the coupon date before the settlement date and the coupon date, following the settlement date
R	$\equiv$	number of days from the settlement date until the coupon date following the settlement date
c	$\equiv$	yearly coupon rate

Arithmetic mean

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\bar{x}$	$\equiv$	arithmetic mean
n	$\equiv$	number of observations
$\sum$	$\equiv$	sum
x_i	$\equiv$	the i-th observation.

Weighted mean

$$\bar{x}_w = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}$$

$\bar{x}_w$	$\equiv$	weighted mean
n	$\equiv$	number of observations
$\sum$	$\equiv$	sum
x_i	$\equiv$	the i-th observation
w_i	$\equiv$	the i-th weight

Standard deviation

$$S = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

S	$\equiv$	standard deviation
n	$\equiv$	number of observations
$\sum$	$\equiv$	sum
x_i	$\equiv$	the i-th observation
$\bar{x}$	$\equiv$	arithmetic mean

Variance

$$V = S^2 \quad \text{or} \quad S = \sqrt{V}$$

V	$\equiv$	variance
S	$\equiv$	standard deviation

Slope of a straight line

$$b = \frac{\text{change in } y\text{-value}}{\text{change in corresponding } x\text{-value}}$$

$b \equiv$ slope of a straight line.

Pearson's correlation coefficient

$$r = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sqrt{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2} \sqrt{n \sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} y_i)^2}}$$

r	$\equiv$	correlation coefficient
x_i	$\equiv$	value of the independent variable for the i-th observation
y_i	$\equiv$	value of the dependent variable for the i-th observation
n	$\equiv$	number of pairs of data points