Study Unit 5 : Calculus

Chapter 6: Sections 6.1, 6.2.1, 6.3.1
Chapter 8: Section 8.1, 8.2 and 8.5

- In Business world the study of change important

  Example: change in the sales of a company; change in the value of the rand; change in the value of shares; change in the interest rate etc.

- Equally important is the *rate* at which these changes take place.

  Example: If the sales of a company increased by R2 000 000,00, it is important to know whether this change occurred over one year, two years or ten years.

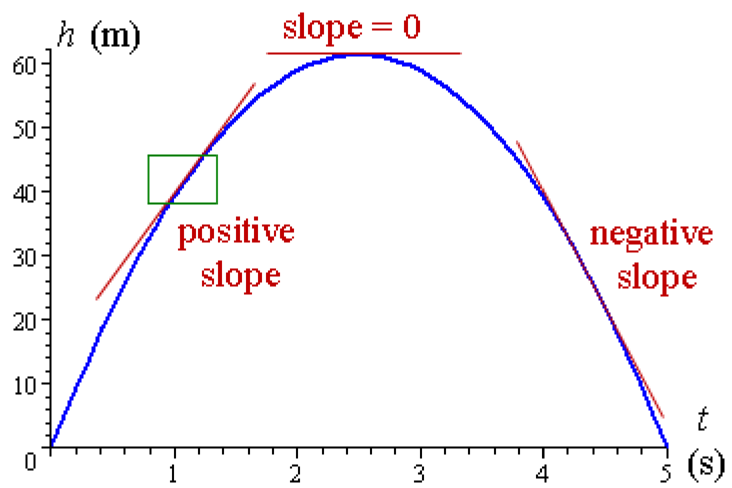
- Rate of change: changes over time, change in costs for different production quantities in a production process, etc.
- Change consists of two components: size and direction.

Let's look at linear function : y = mx + c.

- Slope (m) is the change in *y* which corresponds to change of one unit in the value of *x*: $\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$.

- Slope => indication of rate of change = constant
- Value of the slope => size of the change
- Sign of the slope => direction of the change.
    - positive sign – an increase;
    - negative sign – a decrease.

Let's look at a non-linear function, for example a quadratic function



The size and direction are not constant, but change continuously.

Use the mathematical technique of Differentiation to determine the rate of change.

1. Differentiation

- Slope of a curve = change in *y* / change in *x* = rate curve change
- Use differentiation to get slope at a given point
- d*y*/d*x* = $f'(x)$ derivative of *y* with respect to *x*
- Pronounce this as ”dee-y-dee-x.”
- Use rules of differentiation to obtain the derivative
- Many rules of differentiation –look at just 1 namely

Basic Rule:

If $f(x) = x^n$, then $f'(x)$ or $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$ for $n \neq 0$

For example

- n integer and positive

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^4) = 4x^{4-1} = 4x^3$$

- n integer and negative

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$$

- n is a fraction and positive

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

- n is a fraction and negative

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}} = -\frac{1}{2(x^3)^{\frac{1}{2}}} = -\frac{1}{2\sqrt{x^3}}$$

Note:

1. The derivative of any constant term say *a*, that is a term which consists of a number only, is zero:

   $\frac{da}{dx} = 0$, where *a* is a constant.

   Example: $f(x) = 4$ then $f'(x) = 0$.

2. $\frac{\mathrm{d}}{\mathrm{dx}}\left[\mathrm{a\,f}\left(\mathrm{x}\right)\right] = a f'(x)$.

Example: $f(x) = 7x^5$ then $f'(x) = 7 \times 5x^4 = 35x^4$.

3. If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

Example: $f(x) = 7x^5 + 2x^3$ then $f'(x) = 35x^4 + 6x^2$

Example: $f(x) = 4 + 8x^2$ then $f'(x) = 0 + 16x$

Steps:

1. First we need to **simplify** the given expression so that we can use the basic rule of differentiation.
2. Secondly we differentiate the new expression **using the basic rule** $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ where $n \neq 0$ of differentiation

For example:

1. $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{x^3}) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$
2. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Discussion class example 19, 20

Question 19

Differentiate the following expression:

$$x^3 - 4x^2 + 4x + 5$$

Solution

We can differentiate the expression using the basic rule $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$ where $n \neq 0$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} x^3 - 4x^2 + 4x + 5 &= (1)(3)x^{3-1} - (4)(2)x^{2-1} + (4)(1)x^{1-1} + 0 \\ &= 3x^2 - 8x + 4x^0 \qquad \text{because } x^0 = 1 \\ &= 3x^2 - 8x + 4\end{aligned}$$

Question 20

Differentiate the following expression:

$$x^2(x-4-\sqrt{x})$$

Solution

The basic rule of differentiation states that $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ when $n \neq 0$. To make use of this rule we first need to simplify the expression so that has the same format. We can write $\sqrt{x}$ as $x^{\frac{1}{2}}$ when changing from square root form to exponential form. Thus

$$\begin{aligned}
x^2(x-4-\sqrt{x}) &= x^2(x-4-x^{\frac{1}{2}}) \\
&= x^2x^1 - 4x^2 - x^2x^{\frac{1}{2}} \\
&= x^{2+1} - 4x^2 - x^{2+\frac{1}{2}} \\
&= x^3 - 4x^2 - x^{2\frac{1}{2}} \\
&= x^3 - 4x^2 - x^{\frac{5}{2}}
\end{aligned}$$

Now we differentiate the simplified expression, using the basic rule of differentiation namely $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ when $n \neq 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\; x^3 - 4x^2 - x^{\frac{5}{2}} = 3x^{3-1} - 4(2)x^{2-1} - (\frac{5}{2})x^{\frac{5}{2}-1}$$

$$= 3x^2 - 8x - \frac{5}{2}x^{\frac{3}{2}}$$

$$= 3x^2 - 8x - \frac{5}{2}\sqrt{x^3}$$

- **Application :**
  - Also called rate of change because slope is rate of change
  - Slope of a tangent line at a given point – derivative at that point.
  - Minimum or maximum, vertex, turning point => slope = 0 => dy/dx = 0

  Discussion class example 14 (c) using differentiation

Question 14(c)

$$\text{Profit} = -30P^2 + 7800P - 432\,000$$

? Maximum profit and price

The profit function derived in is a quadratic function with

$$a = -30,\ b = 7800 \text{ and } c = -432\,000.$$

As a < 0 the shape of the function looks like a “sad face” and the function thus has a maximum at the function’s turning point or vertex (P ; Q).

The price P at the turning point, or where the profit is a maximum, is

$$P = -\frac{b}{2a} = -\frac{7800}{2\times -30} = \frac{-7800}{-60} = 130$$

and thus the maximum profit :

$$\text{Profit} = -30(130)^2 + 7800(130) - 432\,000 = 75\,000.$$

Using differentiation:

Now $\text{Profit} = -30P^2 + 7800P - 432\,000$

The slope of the tangent line at the turning point is zero. Determine the slope of the tangent line by differentiating the profit function:

Thus

$$\text{Profit}' = -30(2)P^{2-1} + 7800(1)P^{1-1} - 0$$
$$\text{Slope} = -60P + 7800$$

Now slope = 0 at the turning point:

$$-60P + 7800 = 0$$

$$-60P = -7800$$

$$P = \frac{-7800}{-60}$$

$$P = 130$$

Same as previous method.

- Marginal analysis

$$\text{Marginal revenue} = \frac{\textbf{change in revenue}}{\textbf{change in number of units}}$$

➔ slope of the revenue function.

➔ $MR = \mathrm{d}TR/\mathrm{d}Q$

$$\text{Marginal cost} = \frac{\textbf{change in cost}}{\textbf{change in number of units}}$$

➔ slope of the cost function.

➔ $MC = \mathrm{d}TC/\mathrm{d}Q$

Discussion class example 21

Question 21

What is the marginal cost when Q =10 if the total cost is given by:

$$TC = 20Q^4 – 30Q^2 + 300Q + 200?$$

Solution

The marginal cost function is the differentiated total cost function. Thus by differentiating the total cost function we can determine the marginal cost function. Now if the total cost function is

$$TC = 20Q^4 – 30Q^2 + 300Q + 200$$

then the marginal cost function is

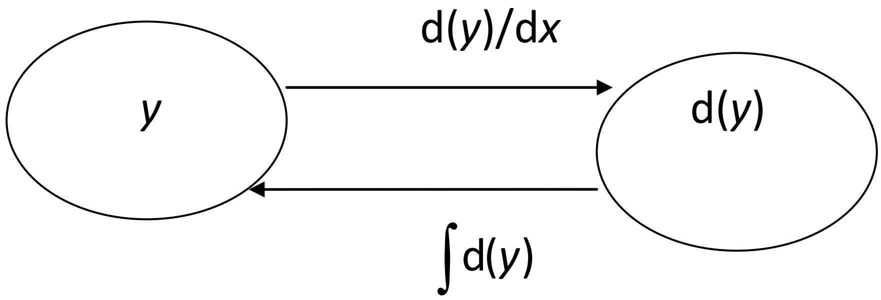
$$MC = \frac{\mathrm{d}TC}{\mathrm{d}Q} = 80Q^3 - 60Q + 300.$$

Now the marginal cost function’s value when Q is equal 10 is

$MC = 80(10)^3 – 60(10) + 300 = 80\ 000 – 600 + 300 = 79\ 700.$

2. Integration

- Is the reverse of differentiation



- Indefinite integral : different rules

  Steps:

- Simplify function before you integrate – write it so that you can apply the integration rule for example
  - $\int(ax+b)=\int ax+\int b$
  - $\int\frac{1}{\sqrt{x}}=\int x^{-\frac{1}{2}}$

- Apply basic integration rule

$$\int x^n \mathrm{d}x=\frac{x^{n+1}}{n+1}+c \quad \textbf{where } n\neq -1$$

$$\int a \ \mathrm{d}x=\frac{ax^{0+1}}{1}+c=ax+c \quad \textbf{where a is a constant}$$

Discussion class example 22 and 23

- Hint : test your answer: differentiate answer, must be equal to function integrated.

Question 22

Evaluate the following

$$\int (x^2 + 2x + 3)\mathrm{d}x$$

Solution

To integrate the function we make use of the basic rule of integration namely $\int x^n = \dfrac{x^{n+1}}{n+1} + c$ when $n \neq -1$. Therefore:

$$\begin{aligned}
\int (x^2 + 2x + 3)\mathrm{d}x &= \int x^2 \ \mathrm{d}x + \int 2x \ \mathrm{d}x + \int 3\mathrm{d}x \\
&= \frac{x^{2+1}}{2+1} + \frac{2x^{1+1}}{1+1} + \frac{3x^{0+1}}{0+1} + c \\
&= \frac{x^3}{3} + \frac{2x^2}{2} + \frac{3x}{1} + c \\
&= \frac{x^3}{3} + x^2 + 3x + c
\end{aligned}$$

Question 23

Determine $\int \frac{Q+1}{\sqrt{Q}}\,\mathrm{d}Q$

Solution

First simplify the function to be integrated:

$$\frac{Q+1}{\sqrt{Q}} = \frac{(Q+1)}{Q^{\frac{1}{2}}}$$

$$= (Q+1)Q^{-\frac{1}{2}}$$

$$= Q^{\frac{1}{2}} + Q^{-\frac{1}{2}}$$

Integrate the function using rule $\int x^n = \frac{x^{n+1}}{n+1} + c$ when $n \neq -1$

$$\int (Q^{\frac{1}{2}} + Q^{-\frac{1}{2}})\;\; \mathrm{d}Q = \int Q^{\frac{1}{2}} + \int Q^{-\frac{1}{2}}$$

$$= \frac{Q^{\frac{1}{2}+1}}{\frac{1}{2}+1} \quad + \quad \frac{Q^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{Q^{\frac{3}{2}}}{\frac{3}{2}} \quad + \quad \frac{Q^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \sqrt{Q^3} \times \frac{2}{3} + \sqrt{Q} \times \frac{2}{1} + c$$

$$= \frac{2\sqrt{Q^3}}{3} + 2\sqrt{Q} + c$$

Definite integral: area under a given curve between two points *a* and *b*:

$$\int_a^b x^n \mathrm{d}x = \frac{x^{n+1}}{n+1}(x=b) \; - \; \frac{x^{n+1}}{n+1}(x=a)$$

- Steps :
    1. Simplify the function
    2. Integrate the function by applying the basic rule of integration
    3. Calculate the value of the integrated function at the value *a* – substitute the values *a* into the integrated function – answer 1
    4. Calculate the value of the integrated function at the value *b* – substitute the values *b* into the integrated function – answer 2
    5. Subtract answer 2 from answer 1

Discussion class example 24

Question 24

Evaluate

$$\int_{-1}^{1} (z+1)\ \mathrm{dz}$$

Solution

To determine a definite integral we first integrate the function, using the basic rule $\int x^n = \dfrac{x^{n+1}}{n+1} + c$ when $n \neq -1$, and then substitute the values between which the integral has to be calculated, into the integrated function.

Step 1: Integrate function:

$$\int_{-1}^{1} (z+1)\ \ \mathrm{dz} = \int_{-1}^{1} (z^1 + z^0)\ \ \mathrm{dz}$$

$$\left(\frac{z^{1+1}}{1+1} + \frac{z^{0+1}}{0+1}\right)\Bigg|_{-1}^{1}$$

$$= \left(\frac{z^2}{2} + z\right)\Bigg|_{-1}^{1}$$

Step 2: Substitute the values between which the integral has to be calculated: Thus

$$F(x)\Big|_{x=b}^{x=a} = F(a) - F(b).$$

Thus

$$\left(\frac{z^2}{2} + z\right)\Bigg|_{-1}^{1} = \left[\frac{z^2}{2} + z\right]_{\text{with } z=1} - \left[\frac{z^2}{2} + z\right]_{\text{with } z=-1}$$

$$= \left[\frac{(1)^2}{2} + (1)\right] - \left[\frac{(-1)^2}{2} + (-1)\right]$$

$$= \left[1\frac{1}{2}\right] - \left[\frac{1}{2} - 1\right]$$

$$= 1\frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$= 1\frac{1}{2} + \frac{1}{2}$$

$$= 2$$