

Study Unit 4 : Non-Linear functions

Chapter 4 : Sections 4.1– 4.4

Types of non-linear functions:

- **Polynomials**

- Linear function : $y = mx + c$ – chapter 2
- Quadratic function : $y = ax^2 + bx + c$
- Cubic function : $y = ax^3 + bx^2 + cx + d$
- n -th order polynomials : $y = ax^n + bx^{n-1} + cx^{n-2} + \dots + \text{constant}$

- **Exponential functions**

$$y = a^x : \text{general format (a = constant)}$$

$$y = 10^x : \text{scientific format}$$

$$y = e^x : \text{natural format}$$

- **Logarithm functions**

$$y = \log_a x$$

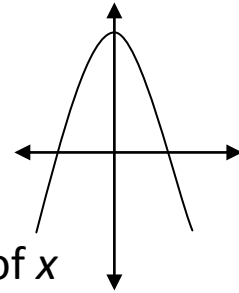
$$y = \log_{10} x = \log x$$

$$y = \log_e x = \ln x$$

- **Hyperbolic functions** $y = \frac{a}{bx + c}$

1. Quadratic function

$$y = ax^2 + bx + c$$

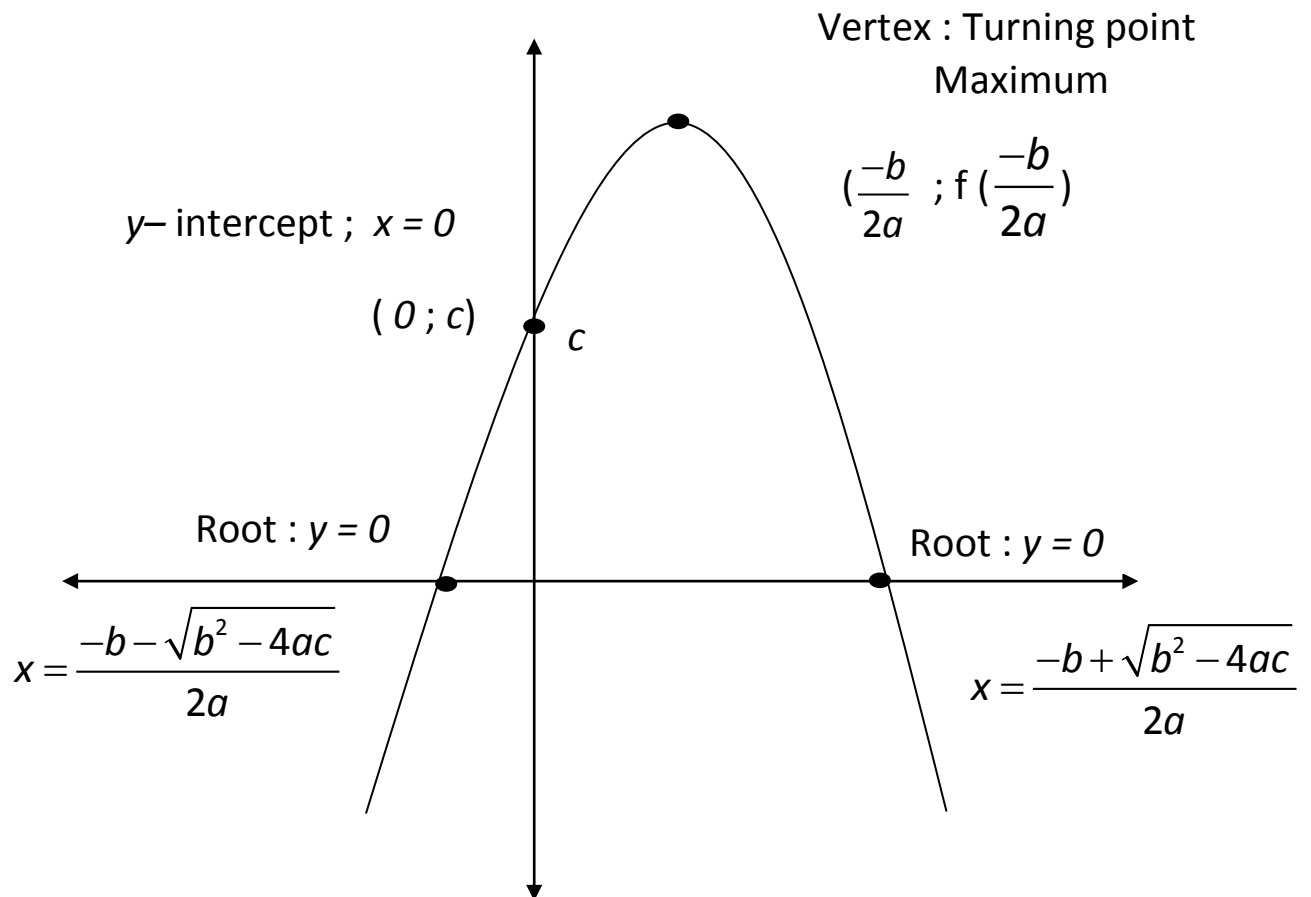


- polynomial of **degree 2** – highest power of x
- c : **y-axis intercept** : cuts y-axis
 - point where $x = 0$: $(0 ; c)$
- a – **shape** of the function
 - $a > 0$: a positive : smiling face
 - $a < 0$: a negative : sad face
- **vertex** : turning point : function maximum or minimum
 - point $(x;y)$ with $x = \frac{-b}{2a}$ and y the function value if

$$x = \frac{-b}{2a} \text{ (substitute the answer for } x \text{ into the function and solve } y\text{)}$$
- **Roots** or x-intercept : cuts x-axis
 - make $y = 0$ and solve x by
 - factorisation or quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example :



Example:

Find the coordinates of the vertex of the graph

$$y = 4x^2 - x - 3.$$

- Vertex = extreme point = turning point = max or min
- $a > 0$ thus graph = smiling face \Rightarrow min exists
- x coordinate of minimum is $x = \frac{-b}{2a}$

Comparing $y = 4x^2 - x - 3$ with $y = ax^2 + bx + c \Rightarrow a = 4$,
 $b = -1$, $c = -3$. Thus

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-1)}{2(4)}$$

$$x = \frac{1}{8} = 0,125$$

- y value if $x = 0,125$ is

$$y = 4x^2 - x - 3$$

$$y = 4(0,125)^2 - 0,125 - 3$$

$$y = -3,0625$$

Coordinates of the vertex of the graph $y = 4x^2 - x - 3$ are
 $(0,125; -3,0625)$

Graph : $y = ax^2 + bx + c$

- Choose random x values and substitute into function to calculate y . Draw $(x ; y)$ coordinate points and graph function

Or

- Calculate the
 - **Roots** or x -intercept
 - **y -intercept** : $(0;c)$
 - turning point or **vertex**

and draw coordinate points and graph of function.

Application:

- **supply and demand; break-even etc.**
- **Maximum or minimum**

Discussion class example 14

Question 14

The demand function for a commodity is $Q = 6\,000 - 30P$. Fixed costs are R72 000 and the variable costs are R60 per additional unit produced.

- (a) Write down the equation of total revenue and total costs in terms of P .
- (b) Determine the profit function in terms of P .
- (c) Determine the price at which profit is a maximum, and hence calculate the maximum profit.
- (d) What is the maximum quantity produced?
- (e) What is the price and quantity at the break-even point(s)?

Solution

- (a) Given are the quantity demanded as $Q = 6000 - 30P$, the fixed costs of R72 000 and the variable costs per unit of R60. Now

$$\text{Total Revenue} = \text{Price} \times \text{Quantity}$$

$$TR = PQ$$

$$TR = P(6000 - 30P)$$

$$TR = 6000P - 30P^2$$

$$\text{Total cost} = \text{Fixed Cost} + \text{Variable Cost}$$

$$TC = 72000 + 60Q$$

$$TC = 72000 + 60(6000 - 30P)$$

$$TC = 72000 + 360000 - 1800P$$

$$TC = 432000 - 1800P$$

- (b) Profit is total revenue minus total cost. Thus

$$\text{Profit} = TR - TC$$

$$= 6000P - 30P^2 - (432000 - 1800P)$$

$$= -30P^2 + 7800P - 432000$$

- (c) The profit function derived in (b) is a quadratic function with

$$a = -30; b = 7800; \text{ and } c = -432000.$$

As $a < 0$ the shape of the function looks like a “sad face” and the function thus has a maximum at the function’s turning point or vertex ($P ; Q$).

The price P at the turning point or where the profit is a maximum is

$$P = -\frac{b}{2a} = -\frac{7800}{2 \times -30} = \frac{-7800}{-60} = 130$$

and thus the maximum profit

$$\text{Profit} = -30(130)^2 + 7800(130) - 432\,000 = 75\,000.$$

- (d) The maximum quantity produced at the maximum price of R130 calculated in (c) is

$$Q = 6000 - 30(130) = 2100.$$

- (e) At break-even the profit is equal to zero. Thus

$$\text{Profit} = -30P^2 + 7800P - 432\,000 = 0.$$

As the profit function is a quadratic function we use the quadratic formula with

$a = -30$ and $b = 7800$ and $c = -432\,000$ to solve P . Thus

$$\begin{aligned}
 P &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-7800 \pm \sqrt{(7800)^2 - 4(-30)(-432000)}}{2 \times -30} \\
 &= \frac{-7800 \pm \sqrt{9000000}}{-60} \\
 &= \frac{-7800 \pm 3000}{-60} \\
 &= \frac{-4800}{-60} \text{ or } \frac{-10800}{-60} \\
 &= 80 \text{ or } 180
 \end{aligned}$$

Now if $P = 80$ then $Q = 6000 - 30(80) = 3600$ and if $P = 180$ then $Q = 6000 - 30(180) = 600$.

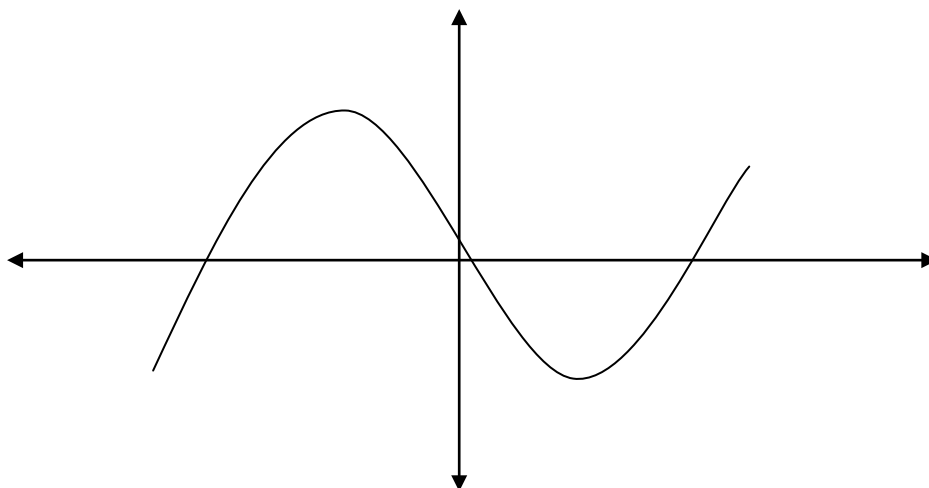
Thus the two break-even points are where the price is R80 and the quantity 3 600, and where the price is R180 and the quantity 600.

2. Cubic function

$$y = ax^3 + bx^2 + cx + d$$

- polynomial of **degree 3** – highest power of x
- 1 or 3 roots
- 0 or 2 turning points

Example:



Graph : Choose random x values and substitute into function to calculate y. Draw (x ; y) coordinate points and graph function

Application:

- supply and demand; break-even etc.
- Maximum or minimum - Differentiation

3. Exponential functions

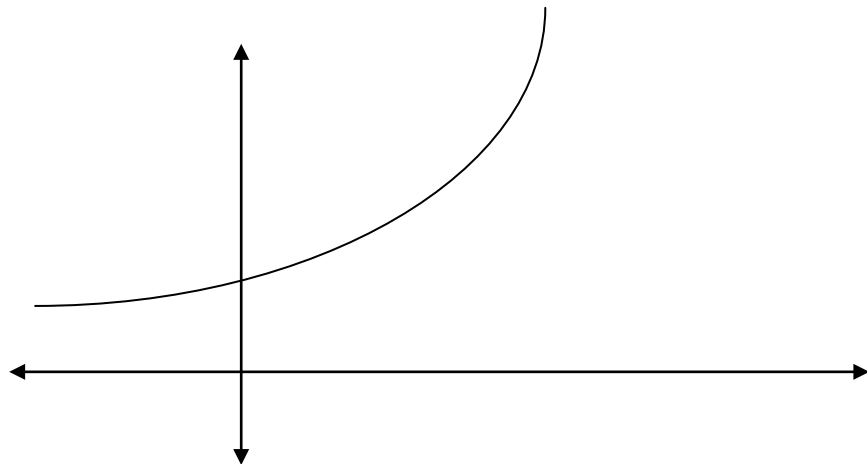
$y = a^x$: general format (a = constant)

$y = 10^x$: scientific format

$y = e^x$: natural format

- a = base = constant
- x = index or power = variable
- e = unending number = 2,7182818....

Example



Properties:

- Continuously pass through point (0 ; 1)
- If power > 0 and $a > 0$, curve increases : growth curve
- If power < 0 and $a > 0$, curve decrease ; decay curve
- If $a > 0$ curve above x-axis
- If $a < 0$ curve below x-axis

Graph : Choose random x values and substitute into function to calculate y. Draw (x ; y) coordinate points and graph function

Rules: Let A and B be any two bases and x and y any two powers then

1. $A^x \times A^y = A^{x+y}$ $2^2 \times 2^3 = 2^{2+3} = 2^5$
2. $A^x \div A^y = A^{x-y}$ $2^4 \div 2^3 = 2^{4-3} = 2^1$
3. $(A \times B)^x = A^x \times B^x$ $(2 \times 3)^3 = 2^3 \times 3^3$
4. $(A/B)^x = A^x / B^x$ $(2 / 3)^3 = 2^3 / 3^3$
5. $(A^x)^y = A^{x \times y}$ $(2^3)^3 = 2^{3 \times 3} = 2^9$

Note: If a is any number

1. $(a)^0 = 1$ but $0^0 = 0$ $4^0 = 1$
2. $(a)^1 = a$
3. $\frac{1}{a^n} = a^{-n}$ $2/x^4 = 2x^{-4}$
4. $\sqrt[n]{a} = a^{\frac{1}{n}}$ $\sqrt{24} = 24^{\frac{1}{2}}$
5. $a^{2x+4} = a^{15}$ (base the same) **then** $2x + 4 = 15$

Discussion class example 15

Application:

Discussion class example 16a

Question 15

Simplify the following expression

$$\left(\frac{4L^2}{L^{-2}}\right)^2$$

Solution

$$\left(\frac{4L^2}{L^{-2}}\right)^2 = (4L^2 \times L^2)^2 \quad \text{since } \frac{1}{a^b} = a^{-b}$$

$$= (4L^{2+2})^2 \quad \text{since } a^b \times a^c = a^{b+c}$$

$$= (4L^4)^2$$

$$= 4^2 L^{4 \times 2} \quad \text{since } (a^a)^b = a^{a \times b}$$

$$= 16L^8$$

Or

$$\left(\frac{4L^2}{L^{-2}}\right)^2 = \left(\frac{4^2 L^4}{L^{-2 \times 2}}\right)^2 = \frac{4^2 L^4}{L^{-4}}$$

$$= 4^2 L^4 \times L^{-4(-1)} \quad \text{since } \frac{1}{a^b} = a^{-b}$$

$$= 4^2 L^4 L^4$$

$$= 4^2 L^8 \quad \text{since } a^b \times a^c = a^{b+c}$$

$$= 16L^8$$

Question 16

An investment in a bank is said to grow according to the following formula:

$$P(t) = \frac{6\,000}{1 + 29e^{-0,4t}}$$

where t is time in years and P is the amount (principle plus interest).

- (a) What is the initial amount invested?
- (b) Determine algebraically the time in years when the amount will be R4 000.

Solution

- (a) Initial means $t = 0$

$$P = \frac{6000}{1 + 29e^{-0,4 \times 0}}$$

Using your calculator's e^x key

$$P = \frac{6000}{30} = 200.$$

(b) If $P = 4\,000$ then

$$4000 = \frac{6000}{1 + 29e^{-0,4t}}$$

$$1 + 29e^{-0,4t} = \frac{6000}{4000}$$

$$1 + 29e^{-0,4t} = \frac{3}{2}$$

Divide nominator and denominator by 2000

$$29e^{-0,4t} = \frac{3}{2} - 1$$

$$29e^{-0,4t} = \frac{1}{2}$$

$$e^{-0,4t} = \frac{1}{2} \div \frac{29}{1}$$

$$e^{-0,4t} = \frac{1}{2} \times \frac{1}{29}$$

$$e^{-0,4t} = \frac{1}{58}$$

$$\ln(e^{-0,4t}) = \ln\left(\frac{1}{58}\right)$$

Take \ln on both sides

$$-0,4t \ln e = \ln\left(\frac{1}{58}\right)$$

$$\ln a^b = b \ln a$$

$$t = \frac{\ln\left(\frac{1}{58}\right)}{-0,4}$$

Since $\ln e = 1$

$t = 10,15110753$ Using your calculator, rounded to 8 decimal places

$t = 10,2$ years

Rounded to one decimal place

4. Logarithmic functions

$$y = \log_a x$$

$$y = \log_{10} x = \log x$$

$$y = \log_e x = \ln x$$

- Logs is the power of a number : $\log 10 = 1$; $\log 100 = 2$
- $\log_{\text{base}} \text{number} = \text{power}$ same as $\text{Number} = \text{base}^{\text{power}}$

$$\log_2 8 = 3 \quad \Rightarrow \quad 8 = 2^3$$

Rules:

$$1. \log (u \times v) = \log u + \log v$$

$$2. \log (u \div v) = \log u - \log v$$

$$3. \log u^j = j \log u$$

$$4. \log_a x = \log x / \log a = \ln x / \ln a$$

Discussion class example 17 + 18

Application:

Discussion class example 16(b)

Question 17

Evaluate $\frac{\log_3 12,34}{\ln \sqrt{12,34}}$

Solution

Since $\log_a b = \frac{\ln b}{\ln a}$ we can write

$$\frac{\log_3 12,34}{\ln \sqrt{12,34}} = \frac{\ln 12,34}{\ln 3} \times \frac{1}{\ln \sqrt{12,34}}$$

Using your calculator, rounded to 3 decimal places

$$= 1,820$$

Question 18

Solve for Q if $\log Q - \log\left(\frac{Q}{Q+1}\right) = 0,8$

Solution

$$\log(Q) - \log\left(\frac{Q}{Q+1}\right) = 0,8$$

$$\log\left(\frac{\frac{Q}{Q}}{\frac{Q}{Q+1}}\right) = 0,8$$

$$\log \frac{a}{b} = \log a - \log b$$

$$\log\left(Q \times \frac{Q+1}{Q}\right) = 0,8$$

$$\log(Q+1) = 0,8$$

$$Q+1 = 10^{0,8} \quad \log_a b = c \text{ can be written as } a^c = b$$

$$Q = 10^{0,8} - 1$$

Using your calculator, rounded to 9 decimal places

$$Q = 5,309573445$$

$$Q = 5,31$$

Rounded to 2 decimal places