Study Unit 3 : Linear algebra

Chapter 3 : Sections 3.1, 3.2.1, 3.2.5, 3.3

Study guide C.2, C.3 and C.4

Chapter 9 : Section 9.1

1. Two equations in two unknowns

- **Algebraically**

Method 1: Elimination

Step 1: Eliminate 1 variable

- —, + one equation or multiple of equation from other equation
- Indication of size of multiple -> number in front of variable

Step 2: Solve variable 1

Step 3: Substitute value of variable1 back into any one of equations and solve variable2

Discussion class example 6(a)

Question 6a

Solve the following set of linear equations by using the elimination method:

y + 2x = 3 –eq(1)

y – x = 2 –eq(2)

Solution

Step 1: Eliminate 1 variable – say y

Subtract eq2 from eq1 and solve x

$$\begin{array}{rcl} y + 2x = 3 & \textbf{or} & y + 2x = 3 \\ -(y - x = 2) & & -y + x = -2 \\ \hline 0 + 3x = 1 & & \end{array}$$

$$x = \frac{1}{3}$$

Step 2: Substitute value of x back into any one of equations and solve y

$$y + 2x = 3$$

$$y + 2\left(\frac{1}{3}\right) = 3$$

$$y = 3 - \frac{2}{3}$$

$$y = 2\frac{1}{3}$$

Method 2: Substitution

Step 1: Change one of the equation so that any variables is the subject of the equation– eq3

Step 2: Substitute eq3 into the unchanged equation and solve first variable

Step 3: Substitute answer step2 into any equation and solve the second variable

Discussion class example 6(b)

Question 6b

Solve the following set of linear equations by using the substitution method:

y + 2x = 3 –eq(1)

y – x = 2 –eq(2)

Solution

Step 1: Make 1 variable subject of an equation

Say y in eq1:

$$y + 2x = 3$$

$$y = 3 - 2x \qquad -\mathbf{eq}3$$

Step 2: Substitute the value of y into other equation

Substitute eq3 into eq2:

Substitute $y = 3 - 2x$ **into** $y - x = 2$

$$(3 - 2x) - x = 2$$

$$-3x = 2 - 3$$

$$-3x = -1$$

$$x = \frac{1}{3}$$

Step 3 : Substitute value of variable into any equation

Substitute $x = \frac{1}{3}$ into eq(1) or eq(2) – choose eq(2)

$$y - x = 2$$

$$y - \frac{1}{3} = 2$$

$$y = 2\frac{1}{3}$$

- **Graphically**

  1. Draw 2 equations – solution intersect

Discussion class example 6(c)

Question 6c

Solve the following set of linear equations graphically

$y + 2x = 3$ –eq(1)

$y - x = 2$ –eq(2)

Solution

Draw 2 lines –> intersecting = solution

Need 2 points to draw a line:

Eq1 : If $x = 0$ then $y + 2(0) = 3$ or $y = 3$ -> $(0 ; 3)$

If $y = 0$ then $0 + 2x = 3$ or $x = 3/2$ -> $(3/2 ; 0)$

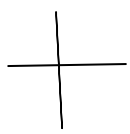
Eq2 : If $x = 0$ then $y - (0) = 2$ or $y = 2$ -> $(0 ; 2)$

If $y = 0$ then $0 - x = 2$ or $x = -2$ -> $(-2 ; 0)$

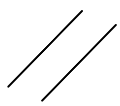
Graph points and draw lines

2. Solutions

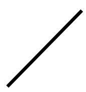
1. 1 Unique – lines intersect



2. No Solution – lines parallel



3. Infinity – lines on top of each other



3. Three equations in three unknowns

Eliminate 1 variable of 3 variables then use method as above. Determine 2 eq's with the same 2 unknowns then use method as above.

Step 1: Write all the equation in the same format – variables one side and values right hand side

Step 2: Eliminate 1 of the 3 variables by + or – one equation from another or one multiply by a value for example:

- eq1 + eq2 = eq4 or
- eq1 – (2 × eq2) = eq4

$\neq x$

$\neq x$

Step 3: Do the same for any other 2 equations –>eq5

Step 4: Now you have 2 equations with the same two variables namely eq4 and eq5 solve as previously explained

Note:

- If one equation has just two variables make one of the variables the subject of the equation–>eq4
- Add or subtract other two eq's that has 3 unknowns –>new equation with 2 variables –> eq5.
- Substitute eq4 into the eq5 and solve your first variable. Etc.

Discussion class example 7

Question 7

Solve the following set of equations

$$x - y + z = 0 \qquad \textbf{(1)}$$

$$2y - 2z = 2 \qquad \textbf{(2)}$$

$$-x + 2y + 2z = 29 \qquad \textbf{(3)}$$

Solution

Step 1: Get 2 eq's with the same 2 unknowns

- eq(2) : already 2 variables
- Add eq(1) and eq(3):

$$\begin{array}{r} x - y + z = 0 \\ -x + 2y + 2z = 29 \\ \hline 0 + y + 3z = 29 \end{array} \qquad \textbf{(4)}$$

Step 2: Solve 2eq with 2 unknowns – any method

- Substitution : Make y the subject of eq4:

$$y = 29 - 3z \qquad \textbf{(5)}$$

- Substitute eq5 into eq2 and solve z:

$$2y-2z=2 \qquad \textbf{(2)}$$

$$2(29-3z)-2z=2$$

$$58-6z-2z=2$$

$$-8z=2-58$$

$$-8z=-56$$

$$z=\frac{-56}{-8}$$

$$z=7$$

Step 3: Substitute z = 7 into eq2

$$2y-2z=2$$

$$2y-2(7)=2$$

$$2y=2+14$$

$$2y=16$$

$$y=8$$

Step 4: Substitute z = 7 and y = 8 into eq1

$$x-y+z=0$$

$$x-8+7=0$$

$$x=1$$

Solution: x = 1; y = 8 and z = 7

2. Applications of simultaneous equations in business

a. Equilibrium market

market equilibrium :

quantity demanded = quantity supplied

price customers willing pay = price producers accept

$$Q_d = Q_s \text{ or } P_d = P_s$$

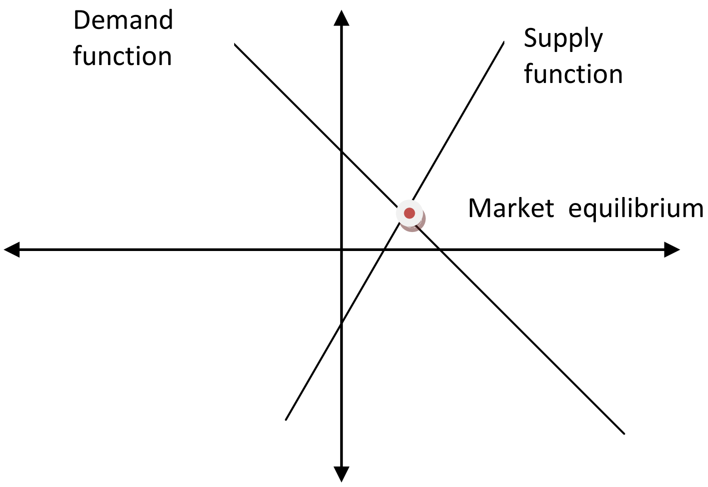
- **Algebraically:**

Solve the demand function and supply function simultaneously . If demand function : $P_d = a - bQ_d$ and supply : $P_s = c + dQ_s$ then

$$a - bQ_d = c + dQ_s$$

- **Graphically:**

Intersection of 2 functions



Discussion class example 8 (a) and (b)

Question 8

In a market we have the following:

Demand function: $Q = 50 – 0,1P$

Supply function: $Q = –10 + 0,1P$

where P and Q are the price and quantity respectively.

(a) Calculate the equilibrium price and quantity.

(b) Draw the two functions, and label the equilibrium point.

Solution

(a) Equilibrium is the price and quantity where the demand and supply functions are equal. Thus determine $Q_d = Q_s$ or

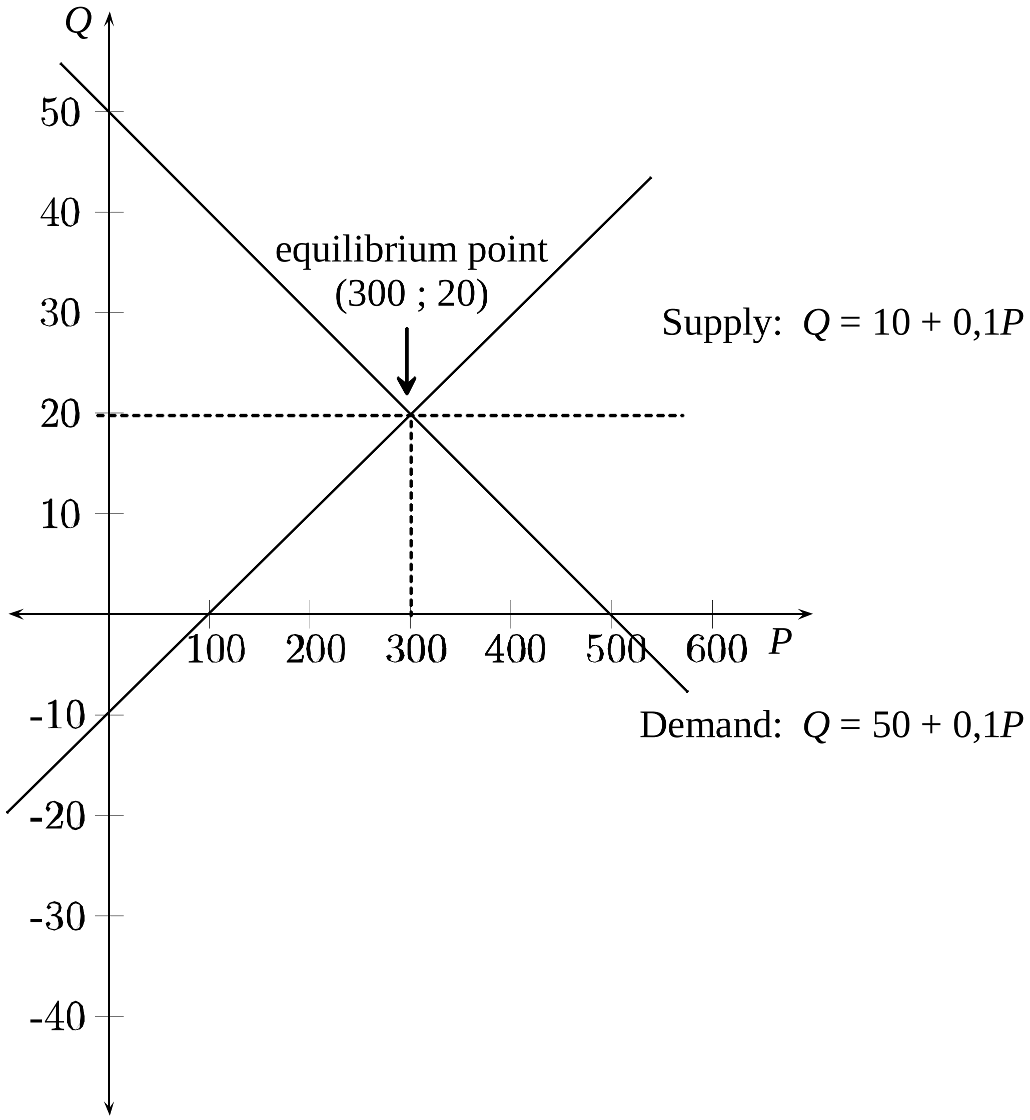
$$\begin{aligned} 50 - 0{,}1P &= -10 + 0{,}1P \\ -0{,}1P - 0{,}1P &= -10 - 50 \\ -0{,}2P &= -60 \\ P &= \frac{-60}{-0{,}2} \\ P &= 300 \end{aligned}$$

To calculate the quantity at equilibrium we substitute the value of P into the demand or supply function and calculate Q. Say we use the demand function then

$$Q = 50 - 0{,}1(300)$$
$$Q = 20$$

The equilibrium price is equal to 300 and the quantity to 20.

(b)



b. Break-even analysis

Do not make a profit or a loss

- Profit = 0
- Profit = revenue – cost = 0
- Total revenue = total cost

Solve :

- Algebraically : solve simultaneous equations

  Revenue = cost

- Graphically : where cost and revenue functions intercepts

Discussion class example 9

Question 9

A company manufactures and sells x toy hand held radios per week. The weekly cost are given by

$$c(x) = 5000 + 2x$$

How many radios should they manufacture to break–even if a radio sells for R202?

Solution

Break-even: Revenue = cost

Revenue = price x quantity = $202x$ and Cost = $5000 + 2x$

$$202x = 5000 + 2x$$

$$202x - 2x = 5000$$

$$200x = 5000$$

$$x = 25$$

OR

Break-even: profit = 0

$$\text{Profit} = \text{Revenue} - \text{cost}$$

$$\text{Profit} = 202x - (5000 + 2x) = 0$$

$$200x - 5000 = 0$$

$$200x = 5000$$

$$x = 25$$

c. Producer and consumer surplus

- **Consumer surplus**

The consumer surplus for demand is the difference between

1. the amount the **consumer is willing to pay** for successive units ($Q = 0$ to $Q = Q_0$) of a product

   and

2. the amount that the **consumer actually paid** for Q_0 units of the product at a market price of P_0 per unit.

CS = Amount willing to pay – Amount actually paid

Example:

Calculate the consumer surplus for the demand function

$P = 50 - 4Q$ when the market price is $P = 10$.

Now:

CS = Amount willing to pay – Amount actually paid

- First we calculate the amount actually paid:

  The number of items the consumer will purchase at a price P = 10 is:

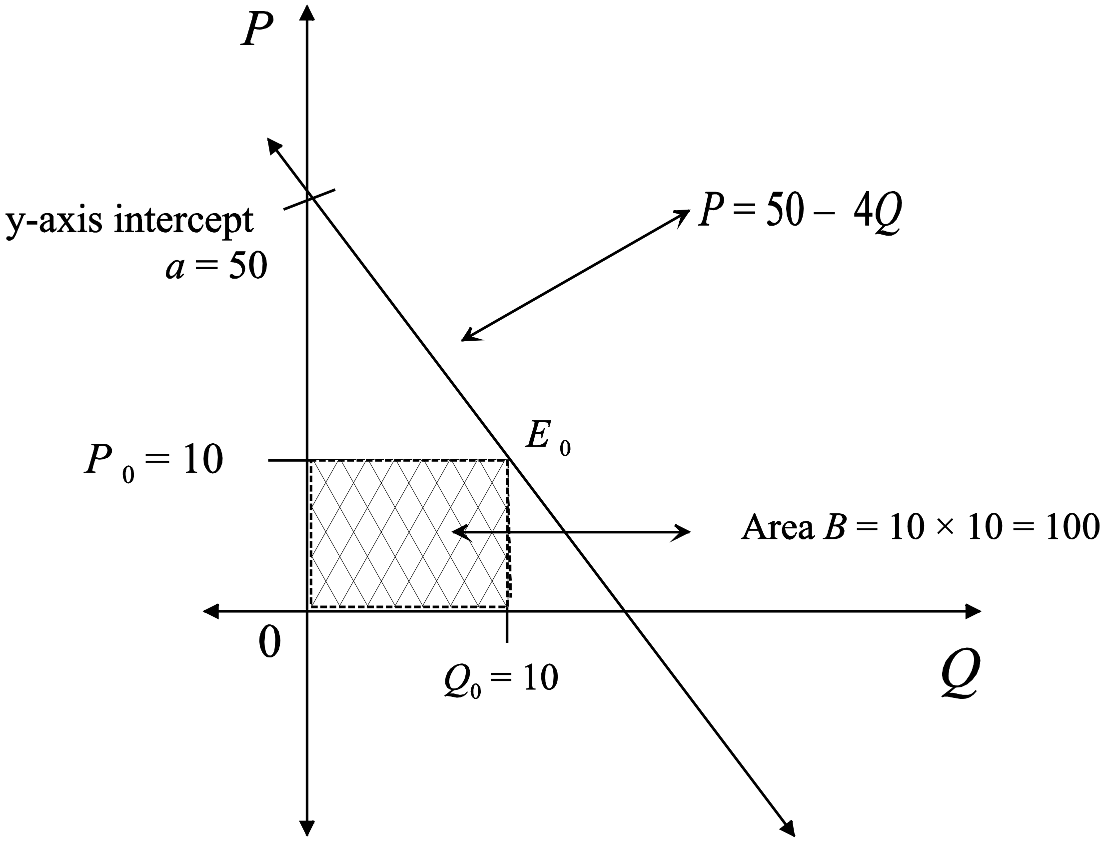
$$
\begin{aligned}
P &= 50 - 4Q \\
10 &= 50 - 4Q \\
10 - 50 &= -4Q \\
-40 &= -4Q \\
\frac{-40}{-4} &= Q \\
10 &= Q \\
Q &= 10.
\end{aligned}
$$

The consumer will buy a quantity of 10 items if the price is R10 per item.

Therefore the consumer will actually spend in total

$P \times Q = 10 \times 10 =$ R100. – (B)

Now graph the demand function and the price and quantity as below:



Graph 1

Now the area B of the rectangle $0P_0E_0Q_0$ with area = length × breadth under the given demand function $P = 50 - 4Q$ is equal to 10 x 10 = 100 which is the same as what the consumer will spend.

Thus graphically the Area B ■ = amount actually spend

- Amount willing to spend

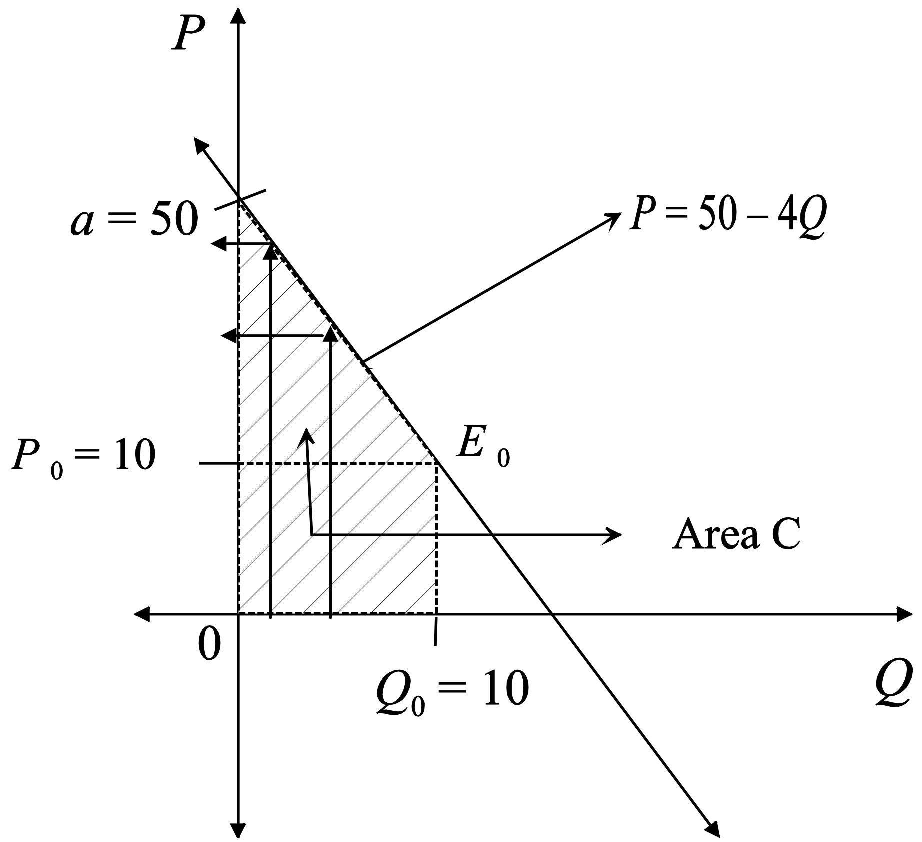
Now the consumer will pay R100 for 10 units. But what will he be willing to pay if the product is scarcer, say only 5 units?

Then

$$P = 50 - 4Q = 50 - (4{\times}5) = \text{R}30 \text{ per unit.}$$

And if the $Q = 2$, then

$$P = 50 - 4Q = 50 - (4{\times}2) = \text{R}42 \text{ per unit.}$$



Graph 2

Thus the total amount which the consumer is thus willing to pay for the first 10 items = the area C under the demand function between $P = 0$ and $P = 10$.

Area (*C*) = area triangle (*A*) plus area square (*B*)

Area (C) = (½ × base × height) + (length × breadth).

$$
\begin{aligned}
\text{Area C} &= \left[\frac{1}{2}\times 10\times(50-10)\right]+(10\times 10)\\
&= \left[\frac{1}{2}\times 10\times 40\right]+100\\
&= \frac{400}{2}+100\\
&= 200+100\\
&= 300
\end{aligned}
$$

The total amount which the consumer is willing to pay for the first 10 units is thus R300

Now the consumer surplus is defined as:

CS = amount willing to pay – amount actual pay

= 300 – 100

= 200.

Alternatively to summarise:

CS = amount willing to pay – amount actual pay

CS = area (C) in graph 1 – area (B) in graph 2

= (area of ◣ + area of ■) – area of ■

= (A) + (B) – (B) = (A)

= area of triangle ◣

= ½ × height × base

= ½ × (50 –10) × (10)

= 200 (same as calculated earlier).

In general:

If you need to determine the demand surplus for a demand function of $P = a - b\,Q$ then the consumer surplus can be calculated by calculating an area of the triangle P_0E_0a which is equal to

½ × height × base

$= ½ \times (a - P_0) \times (Q_0 - 0)$

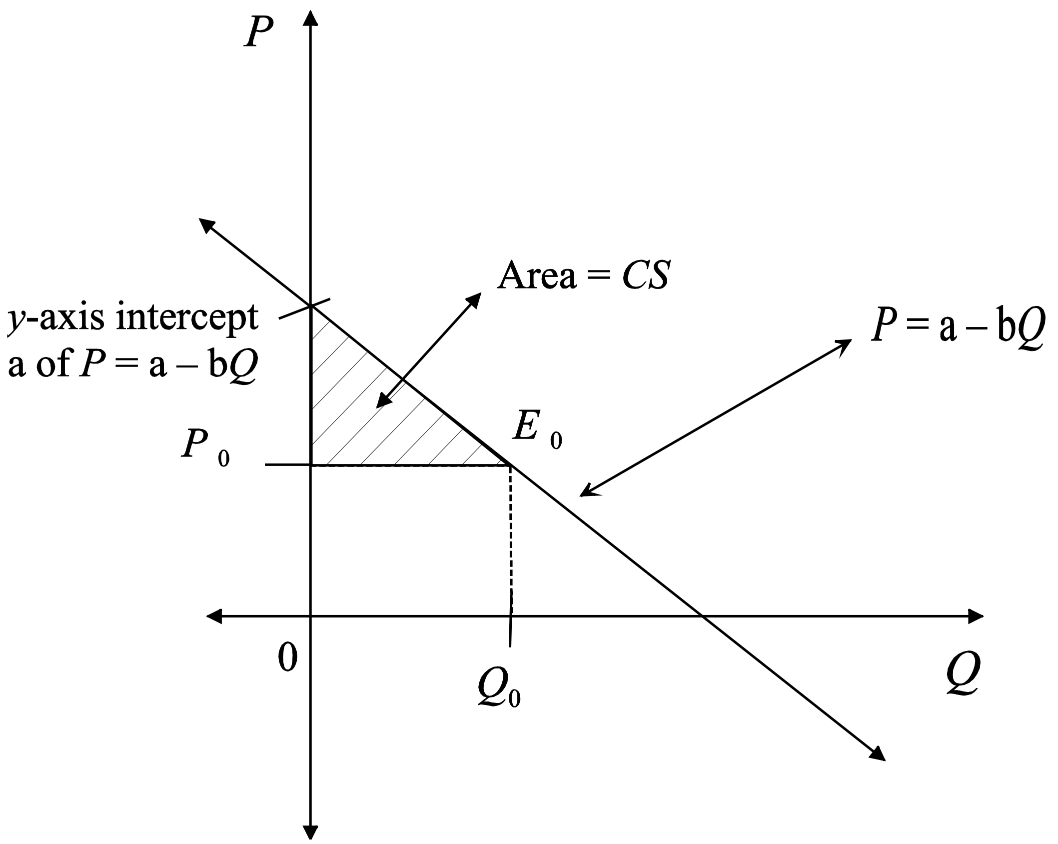
$= ½ \times (a - P_0) \times (Q_0)$

with

- P_0 the value given to you as the market price
- Q_0 is the value of the demand function if $P = P_0$
  (Substitute P_0 into the demand function and calculate Q_0)
- a is the y-intercept of the demand function
  $P = a - bQ$ also known as the value of P if $Q = 0$, or the place where the demand function intercepts the y-axis.

Method:

1. Draw a rough graph of the demand function
2. Calculate Q_0 if P_0 is given and
3. Get the value of a from demand function
4. Calculate the area or CS = ½ × $(a - P_0) \times (Q_0)$



Discussion class example 10

Question 10

Calculate the consumer surplus for the demand function
P = 60 – 4Q when the market price is P = 12.

Solution

- Calculate Q if P = 12

$$P = 60 - 4Q$$

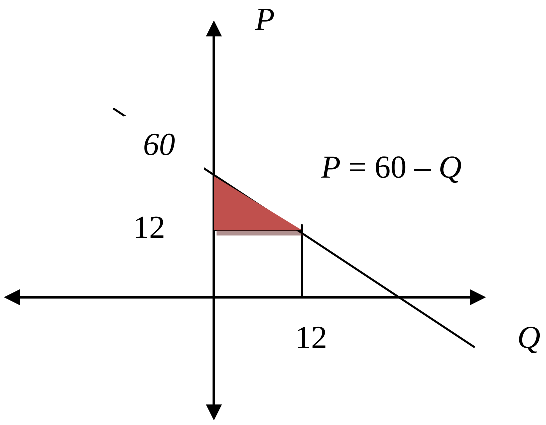
$$12 = 60 - 4Q$$

$$4Q = 60 - 12$$

$$4Q = 48$$

$$Q = 12$$

- Draw a rough sketch of graph



- Calculate the consumer surplus

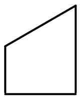
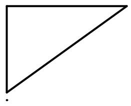
$$\textbf{CS} = \frac{1}{2} \times \textbf{base} \times \textbf{height}$$

$$= \frac{1}{2} \times \mathbf{(}12\mathbf{)} \times \mathbf{(}60 - 12\mathbf{)}$$

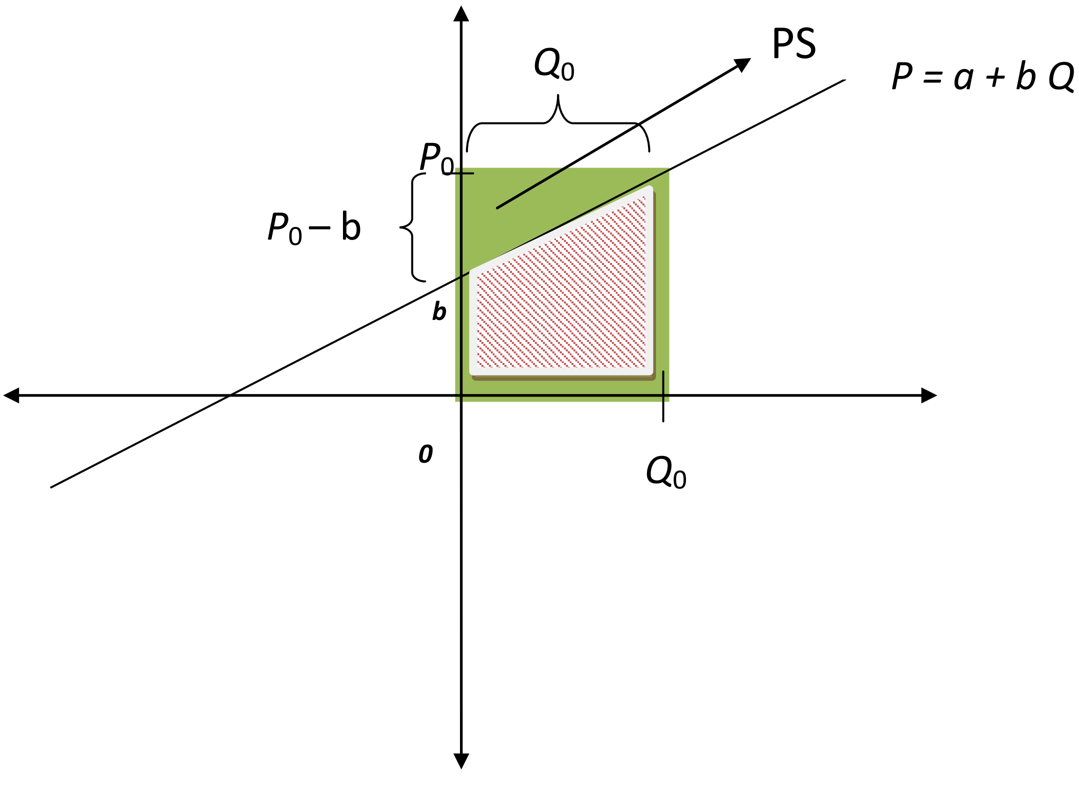
$$= 288$$

- **Producer surplus**

PS = revenue producer receives at Q_0 – revenue producer willing to accept between 0 and Q_0

Producer surplus =  –  = 

= ½ x Q_0 x (P_0 – b)



4. Linear inequalities

(a) Graphics of linear inequality – study guide C.2

Example: $2x + y \leq 120$

Step 1: Change $\geq$ or $\leq$ or $>$ or $<$ to $=$ and draw the graph of the line.

Step 2: Determine region inequality true and colour area

- Substitute a point on either side into inequality and see which point makes the inequality true

(b) Solving a system of inequalities – study guide C.3

Draw each inequality as above (step 1 and step 2)

Select area true for whole system – area simultaneously coloured – **feasible region**

Discussion class example 11 (a) + (b)

See (c) later

Question 11

(a) Draw the lines representing the following constraints:

$$2x + y \leq 120 - (1)$$
$$x + 2y \leq 140 - (2)$$
$$x + y \leq 80 - (3)$$
$$x_1, x_2 \geq 0 - (4)$$

(b) Show the feasible region.

(c) Determine the maximum value of

$$P = 20x + 30y$$

subject to the constraints above.

Solution

Step 1:

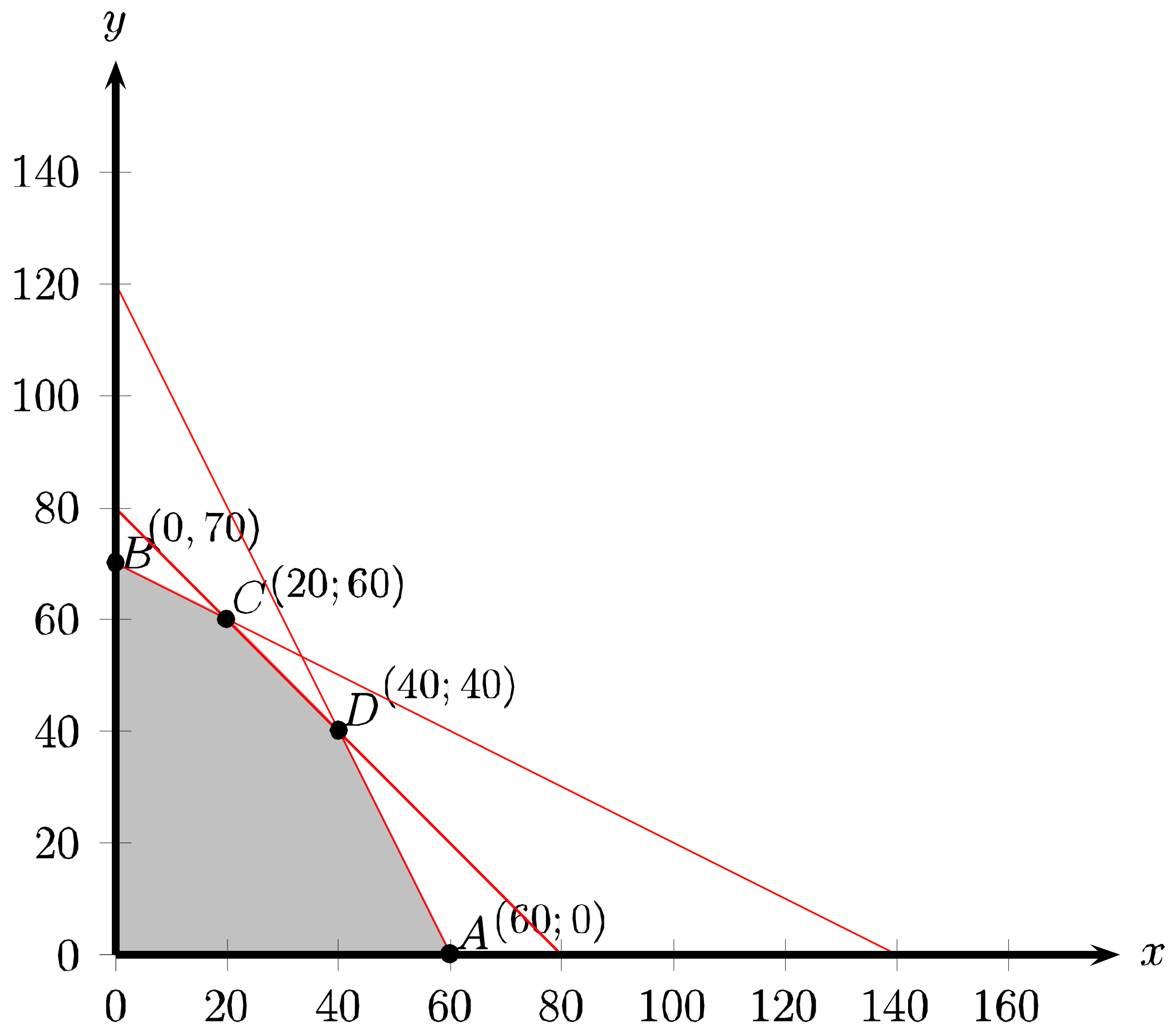
Change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and graph of the line.

Step 2:

Determine the feasible region – substitute a point on either side of the equality

	Step 1		Step 2
Inequality	***y*-axis intercept** **Point (*x* ; *y*) if *x* = 0**	***x*-axis intercept** **Point (*x* ; *y*) if *y* = 0**	**Inequality region**
$2x + y \leq 120$ (1)	$2x + y = 120$ $2(0) + y = 120$ $y = 120$ Point : (0 ; 120)	$2x + y = 120$ $2x + 0 = 120$ $x = \frac{120}{2} = 60$ Point : (60 ; 0)	Select points (0,0) below the line $2(0) + 0 \leq 120$ –True Area below the line
$x + 2y \leq 140$ (2)	$x + 2y = 140$ $0 + 2y = 140$ $y = \frac{140}{2} = 70$ Point (0 ; 70)	$x + 2y = 140$ $x + 2(0) = 140$ $x = 140$ Point (140 ; 0)	Select points (0,0) to left of the line $0 + 2(0) \leq 140$ –True Area to the left of the line
$x + \mathrm{y} \leq 80$ (3)	$x + \mathrm{y} = 80$ $0 + y = 80$ $y = 80$ Point (0 ; 80)	$x + \mathrm{y} = 80$ $x + 0 = 80$ $x = 80$ Point (80 ; 0)	Select points (0,0) below the line $0 + 0 \leq 80$ – True Area below the line
$x, y \geq 0$			Area above the *x*-axis and to the right of the *y*-axis

Step 3: Draw graph and show feasible region



(c) See later

Application of system of linear inequalities in business

1. Linear programming

- Real life – find "best" value under certain conditions – optimisation

- Need to find maximum (profit) or minimum (cost) subject to certain constrains for example resources: labour or materials.

- LP is the problem of maximising or minimising a linear function (profit or cost) called the **objective function** subject to linear **constraints** expressed as inequalities or equations.

- **Formulating a LP problem:**
  - Define the decision variables for example *x* and *y*
  - summarise the information given in a table with the headings:
    - resources (things that have restrictions on),
    - the variables (*x* and *y*) and
    - capacity (amount or number available of the resources).

- **Discussion class example 12 and 13**

Question 12

A manufacturing plant makes two types of inflatable boats: a two-person boat and a four-person boat.

Each two-person boat requires 0,9 labour hours from the cutting department and 0,8 labour hours from the assembly department.

Each four-person boat requires 1,8 labour hours from the cutting department and 1,2 labour hours from the assembly department.

The maximum hours available for the cutting and assembly departments are 864 and 672 respectively.

The company makes a profit of R2 500 on a two-person boat and R4 000 on a four-person boat.

Formulate the constraints and objective function of the linear programming problem if the company would like to know how many two-person boats and four-person boats respectively must the company manufactures to maximise its profit.

Solution

First we define the variables.
Let *x* and *y* be the number of two-person boats and four-person boats made respectively.

Secondly to help us with the formulation, we summarise the information given in a table with the headings:

- resources (items with restrictions),
- the variables (*x* and *y*) and
- capacity (amount or number of the resources available).

	x	***y***	
Resource	**Two-person**	**Four-person**	**Capacity**
Cutting	0,9	1,8	864
Assembly	0,8	1,2	672
Profit	2500	4000	
Number of boats			Never negative

Using the table, the following constraints can be defined:

$$0{,}9x + 1{,}8y \leq 864$$

$$0{,}8x + 1{,}2y \leq 672$$

$$x, y \geq 0$$

As the company would like to maximise their profit, the objective function can be written as:

$$2\,500x + 4\,000y.$$

Question 13

Giapetto’s Woodcarving manufactures two types of wooden toys: soldiers and trains.

- A soldier sells for R27 and uses R10 worth of raw materials.
- A train sells for R21 and uses R9 worth of raw materials.
- The manufacturer of wooden soldiers and trains requires two types of skilled labour: carpentry and finishing.
- A soldier requires 2 hours of finishing and 1 hour of carpentry
- A train requires 1 hour of finishing and 1 hour of carpentry.
- Each week, only 100 finishing hours and 80 carpentry hours are available at most.
- The demand for trains is unlimited, but at most 40 soldiers are bought each week.
- Giapetto’s has a weekly budget of R10 000 for the raw material.

If x is the number of soldier toys made per week and y is the number of train toys made per week, formulate the linear constraints that describe Giapetto’s situation and write down the revenue function (objective function) if his objective is to maximise his revenue.

Solution

First we define the variables. Let *x* be the number of soldier toys manufactured per week and *y* the number of train toys manufactured per week. To help us with the formulation we summarise the information given in a table with the headings: resources (things that have restrictions on), the variables (*x* and *y*) and capacity (amount or number available of the resources).

	x	***y***	
Resource	**soldier**	**train**	**Capacity**
Carpentry	1	1	80
Finishing	2	1	100
Raw materials budget	10	9	10 000
Sales / Revenue	27	21	
Maximum per week	40	-	
Number of toys			Always positive

Using the table the following constraints can be defined:

$x + y \leq 80$	Carpentry
$2x + y \leq 100$	Finishing
$10x + 9y \leq 10\,000$	Budgetary constraint raw materials
$x \leq 40$	Maximum demand per week
$x, y \geq 0$	Non-negativity

As he would like to maximise his revenue and as revenue is equal to quantity times the price or sales, the objective function can be written as 27x + 21y.

- **Solving a LP graphically:**

  Many methods example simplex method and graphics

  Use graphical method in DSC1520.

Step 1 : Draw all the inequalities

Step 2: Determine the feasible region

Step 3: Determine coordinates of all corners of feasible region by substitution or read from graph

Step 4: Substitute corner points into objective function

Step 5: Choose corner point that result in highest/maximisation or lowest/minimisation objective function value.

Discussion class example 11 (c)

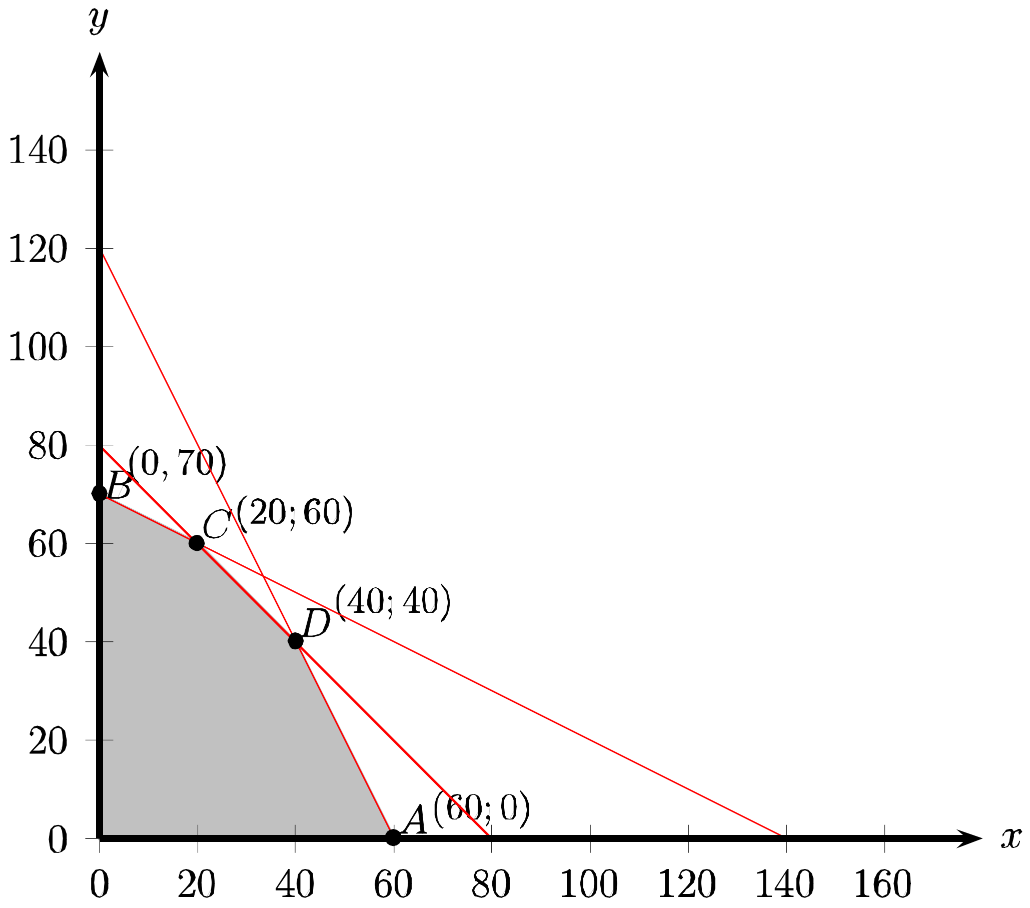
Question 11(c)

Determine the maximum value of $P = 20x + 30y$ subject to the constraints of (a) :

$$2x + y \leq 120 - (1)$$
$$x + 2y \leq 140 - (2)$$
$$x + y \leq 80 - (3)$$
$$x_1, x_2 \geq 0 - (4)$$

Solution:

Determine all the corner points of the feasible region and substitute them into the objective function (function you want to maximise or minimise) and determine the maximum value.



Corner points of feasible region	**Value of $P = 20x + 30y$**
A: $x = 60$; $y = 0$	$P = 20(60) + 30(0) = 1\ 200$
B: $x = 0$; $y = 70$	$P = 20(0) + 30(70) = 2\ 100$
C: $x = 20$; $y = 60$	$P = 20(20) + 30(60) = 2\ 200$ ← **Maximum**
D: $x = 40$; $y = 40$	$P = 20(40) + 30(40) = 2\ 000$
Origin: $x = 0$; $y = 0$	$P = 20(0) + 30(0) = 0$

Maximum P is at point C where $x = 20$; $y = 60$ and $P = 2\ 200$.