Foundation of proofs

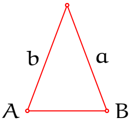
Jim Hefferon

http://joshua.smcvt.edu/proofs

The need to prove

In Mathematics we prove things

'The base angles of an isoceles triangle are equal' seems obvious to a person with mathematical aptitude.



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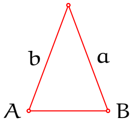

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Another example of a statement that seems obvious to such a person is 'each positive integer factors into a product of primes'.

But is the Pythagorean Theorem 'in a right triangle the square of the length of the hypoteneuse is equal to the sum of the squares of the other two sides' perfectly clear? Does it not require an argument?

A characteristic of our subject is that we show that new results follow logically from those already established.

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- At first we may guess that the polynomial $n^2 + n + 41$ outputs only primes.

n	0	1	2	3	4	5	6	7
$n^2 + n + 41$	41	43	47	53	61	71	83	97

  However, that pattern breaks down; for $n = 41$ the output $41^2 + 41 + 41$ is clearly divisible by 41.

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- When decomposed, $18 = 2^1 \cdot 3^2$ has an odd number $1 + 2$ of prime factors, while $24 = 2^3 \cdot 3^1$ has an even number $3 + 1$ of them. We say that 18 is of *odd* type and 24 is of *even* type.

n	1	2	3	4	5	6	7	8	9
type	even	odd	odd	even	odd	even	odd	odd	even

  Pòlya conjectured that below any $n > 1$ the even types do not outnumber the odd types. The numerical evidence is strong — the statement holds until 906 150 257 — but that number gives a counterexample.

Elements of logic

Propositions

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These are not propositions: ‘$3 + 5$’ and ‘x is not prime.’

Negation

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'It is not the case that $3 + 3 = 6$' is false.

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So the truth value of 'not P' depends only on the truth of P. We say 'not' is a unary logical operator or a unary boolean function since it takes one input, a truth value, and yields as output a truth value.

Conjunction, disjunction

A proposition consisting of the word and between two sub-propositions is true if the two halves are true.

'$3 + 1 = 4$ and $3 - 1 = 2$' is true

'$3 + 1 = 4$ and $3 - 1 = 1$' is false

'$3 + 1 = 5$ and $3 - 1 = 2$' is false

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A compound proposition constructed with or between two sub-propositions is true if at least one half is true.

'$2 \cdot 2 = 4$ or $2 \cdot 2 \neq 4$' is true

'$2 \cdot 2 = 3$ or $2 \cdot 2 \neq 4$' is false

'$2 \cdot 2 = 4$ or $3 + 1 = 4$' is true

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'$2 \cdot 2 = 3$ or $2 \cdot 2 \neq 4$' is false

'$2 \cdot 2 = 4$ or $3 + 1 = 4$' is true

So 'and' and 'or' are binary logical operators.

Truth Tables

Write $\neg P$ for 'not P', $P \wedge Q$ for 'P and Q', and $P \vee Q$ for 'P or Q'. We can describe the action of these operators using truth tables.

P	$\neg P$
F	T
T	F

P	Q	$P \wedge Q$	$P \vee Q$
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

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One advantage of this notation is that it allows formulas of a complexity that would be awkward in a natural language. For instance, $(P \vee Q) \wedge \neg(P \wedge Q)$ is hard to express in English.

Sometimes we prefer using 0 for F and 1 for T. One reason for the preference is that on the left side of the tables the rows make the ascending binary numbers.

P	$\bar{P}$
0	1
1	0

P	Q	$P \cdot Q$	$P + Q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

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0	1	0	1
1	0	0	1
1	1	1	1

In this context ‘not P’ is symbolized $\bar{P}$. Note that $\bar{P} = 1 - P$.

The table makes clear why ‘P and Q’ is symbolized with a multiplication dot $P \cdot Q$.

For ‘P or Q’ the plus sign is a good symbol because ‘or’ accumulates the truth value T.

Other operators: Exclusive or

Disjunction models sentences meaning ‘and/or’. In contrast, ‘Live free or die’, ‘Eat your dinner or no dessert’, and ‘Give me the money or the hostage gets it’ all mean one or the other, but not both.

P	Q	P XOR Q
F	F	F
F	T	T
T	F	T
T	T	F

Other operators: Implies

We model ‘if P then Q’ this way.

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Here P is the *antecedent* while Q is the *consequent*.

Other operators: Bi-implication

Model ‘P if and only if Q’ with this.

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

Mathematicians sometimes write ‘iff’.

All binary operators

We can lists all of the binary logical operators.

P	Q	P α_0 Q
F	F	F
F	T	F
T	F	F
T	T	F

P	Q	P α_1 Q
F	F	F
F	T	F
T	F	F
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...

P	Q	P α_{15} Q
F	F	T
F	T	T
T	F	T
T	T	T

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F	F	T
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These are the unary ones.

P	β_0P
F	F
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P	β_1P
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P	β_2P
F	T
T	F

P	β_3P
F	T
T	T

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A zero-ary operator is constant so there are two: T and F.

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P	Q	R	$P \to Q$	$P \to R$	$(P \to Q) \wedge (P \to R)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	F
T	F	T	F	T	F
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Tautology, Satisfiability, Equivalence

A formula is a *tautology* if it evaluates to T for every value of the variables. A formula is *satisfiable* if it evaluates to T for at least one value of the variables.

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Two propositional expressions are logically equivalent if they give the same input-output relationship. Check that the expressions E_0 and E_1 are equivalent by using truth tables to verify that $E_0 \leftrightarrow E_1$ is a tautology.

For instance, $P \wedge Q$ and $Q \wedge P$ are equivalent. Another example is that $P \rightarrow Q$ and $\neg Q \rightarrow \neg P$ are equivalent.

Non-obvious lines in the implication table

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Our definition of implies takes ‘if Babe Ruth was president then $1 + 2 = 4$’ to be a true statement, because its antecedent is false. Similarly we take ‘if Mallory reached the summit of Everest then $1 + 2 = 3$’ to be true because its consequent is true. Why define implication this way?

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is true for all $n \in \mathbb{N}$.

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Points about implication

P	Q	$P \to Q$
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- As noted on the prior slide, the antecedent P need not be materially connected to the consequent Q.

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- Also noted there are: (1) if the antecedent P is false then the statement as a whole is true, said to be vacuously true and (2) if the consequent Q is true then the statement as a whole is true.

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- Truth tables show that $P \to Q$ is logically equivalent to $\neg(P \wedge \neg Q)$, to $\neg P \vee Q$, and also to the contrapositive $\neg Q \to \neg P$.
- On a table in front of you are four cards, marked 'A', 'B', '0', and '1'. You must verify the truth of the implication, 'if a card has a vowel on the one side then it has an even number on the other.' How to do it, turning over the fewest cards? (This is the *Wason test*; fewer than 10% of Americans get it right.)

Predicates, Quantifiers

The statement

$$\text{`if } n \text{ is odd then } n \text{ is a perfect square'} \qquad (*)$$

involves two clauses, 'n is odd' and 'n is square'. For each the truth value depend on the variable. A predicate is a truth-valued function. An example is the function Odd that takes an integer as input and yields either T or F, as in $\text{Odd}(5) = T$. Another example is Square, as in $\text{Square}(5) = F$.

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A mathematician stating $(*)$ would mean that it holds for all n. We denote 'for all' by $\forall$ so the statement is formally written $\forall n \in \mathbb{N}\big[\text{Odd}(n) \to \text{Square}(n)\big]$. (It is of course a false statement.)

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A quantifier describes for how many values of the variable the clause must be true, in order for the statement as a whole to be true. Besides 'for all' the other common quantifier is 'there exists', denoted $\exists$. The statement $\exists n \in \mathbb{N}\big[\text{Odd}(n) \to \text{Square}(n)\big]$ is true.

Examples of statements written formally, with explicit quantifiers.

- Every number is divisible by 1.

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$$\exists n_0, \ldots, n_4 \in \mathbb{N} \left[(n_0 \neq n_1) \wedge (n_0 \neq n_2) \wedge \cdots \wedge (n_3 \neq n_4) \right.$$
$$\left. \wedge \exists a_0 \in \mathbb{N}(2^{n_0} - 7 = a_0^2) \wedge \cdots \wedge \exists a_4 \in \mathbb{N}(2^{n_4} - 7 = a_4^2) \right]$$

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- Any two integers have a common multiple.

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- The function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$.

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R} \left[(|x - a| < \delta) \to (|f(x) - f(a)| < \varepsilon) \right]$$

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is

$$\exists n \in \mathbb{N} \neg\left[\mathrm{Odd}(n) \rightarrow \mathrm{Square}(n)\right]$$

which is equivalent to this.

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Thus a person could show that 'every odd number is a perfect square' is false by finding a number that is both odd and not a square.

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Simililarly the negation of a '$\exists$' statement is a '$\forall\neg$' statement.